Introduction to Groundwater Hydrology

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This is a course note written & assembled by Ye Zhang for an introductory Groundwater Hydrology class.

Spring 2011
GEOL 4444/ GEOL 5444
4 CREDITS
GRADING: A-F
Prerequisite: Calculus I & II

Location: GE318
Class days & times: Tues + Thurs (9:35 am ∼ 10:50 am)
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NOTE: The lecture notes will be periodically posted on the Wyoweb course website, usually before a class so students can familiarize themselves with the martials. Please make a habit of checking for notes and other materials from the site.

NOTE: The lecture notes do not include: (1) solutions to the exercises and homework; (2) proofs to theories and equation derivations. These will be presented during the lectures only. Thus, do not rely on the notes for everything — attendance and in-class participation are key. Due to time limitation, not all proofs/derivations can be presented. Some advanced derivations (though noted as "details given in class" in notes) will be posted on Wyoweb, typically under a folder called "Advanced Folder".

0.1 This Class

In this class, a fairly rigorous mathematical treatment is presented, which deviates from the typical introductory classes that emphasize the more applied aspects. I feel that mastering of such skills would come naturally after a student has first developed a firm grasp of the fundamentals. This course is thus designed at the upper undergrad and graduate level, appropriate for the level of mathematical rigor contained herein. To fully comprehend the materials presented, a student should have a sufficient knowledge in college math, i.e., Calculus (and preferably differential equations). If in doubt, please review Chapter 1, Basic Math to determine if a preliminary class may be necessary before taking this class.

Throughout the course, many formulations and equations are developed using differential equations and integration. **The emphasis is on understanding how these equations are obtained and their underlying assumptions.** However, you will rarely be tested on equation derivations in exams/quizes (those few that you may be tested on — I'll let you know), nor is it necessary to memorize a large number of specific formulas or solutions (typically, the exam/quiz will provide the necessary formulas so understanding what these mean and how to use them is key). Therefore, do not *unduly* dwell upon the derivation details, almost all homework and exam questions can be solved using a pencil and a calculator.

0.1.1 Textbook

The textbook for this course is **Groundwater Science** by C. R. Fitts, published in 2002 by the Academic Press. Many exercises, homework, and reading assignments are selected from this book. It is recommended that you own a copy. For some homework problems, the book provides the final answers (on page 415). However, your completed assignment must provide the entire analysis rather than a single number. Some of the answers occasionally contain typographical errors, so do not entirely rely on them to judge your own results. There are a few other minor mistakes in the current version of the book. See an Errata list:

http://www.academicpress.com/groundwater.

Make sure you make the appropriate corrections before using the textbook.

To allow an in-depth analysis of the steady state and transient flow systems within a one-semester time frame, the chapters in this book on water chemistry and groundwater contaminations are not included. However, your money should not be wasted as these materials can be useful in subsequent classes on groundwater modeling. To supplement the textbook, additional materials are compiled & assembled based on several books, each with its own special emphasis:

- General Overview: *Groundwater*, Freeze & Cherry, 1979, Prentice Hall, p 604.

- General Overview: *Applied Hydrogeology*, Fourth Edition, C. W. Fetter, 2001, Prentice Hall, p 598.

- Practical Problems: *Practical Problems in Groundwater Hydrology*, Bair & Lahm, 2006, Prentice Hall.

- Mathematical Treatment: *Dynamics of Fluids in Porous Media*, J. Bear, 1988, Dover Publications, p 784.

- Mathematical Treatment: *Quantitative Solutions in Hydrogeology and Groundwater Modeling*, Neven Kresic, 1997, Lewis Publishers, p 461 (it addresses a variety of analytical problems with clear graphics: an excellent self-study book after you take this class).

- Mathematical Treatment: *Groundwater Hydraulics*, (Chinese) Y-Q Xue et. al (Editors), 1986, Geology Publishing House of P.R. China, p 345.

Other materials are obtained from course notes prepared by other professors (references are listed in the notes).

Due to time limitation, this course cannot hope to cover every aspect of the subject as presented in the books. For example, materials on Groundwater Management, Groundwater Chemistry, Solute Transport, Vadose-Zone Hydrology, and Field Methods are not covered. Some of these topics can be understood by independent study, others may be the contents of more advanced or specialized classes. Thus, most of this class is devoted to the study of **single-phase** (water), **uniform-density** flow moving through **non-deforming** porous media (e.g., groundwater aquifers that are not going through compaction or subsidence though we may briefly touch on these topics). The relevant physical laws and mathematical equations are developed in detail. Immiscible, multiphase fluids are not covered. Note, however, the basic approaches in formulation and derivation are similar.

0.1.2 Tools

For this class, besides some simple Excel exercises, we will not generally do computer-aided modeling analysis (this will be formally taught in a separate class I teach on groundwater modeling). For some labs, a personal laptop might be ideal to facilitate repetitive calculations.

Since theoretical vigor in the mathematical development is emphasized, I expect that students develop a good understanding on the fundamental aspects of the topics. Besides a few important formulations (where memorizing them will serve you well), you're not expected to memorize a large number of equations. Most homework and exercises can be solved with pencil, ruler, and a calculator. Make sure you have them during both lectures and exams.

0.1.3 Questions and Answers

Students can ask questions: (1) during office hour; (2) during lectures. As a rule, email is a last resort since I receive a lot of them every day and your message stands a chance of being overlooked by mistake.

0.1.4 Homework, Labs, Exams and Grades

When working on homework/lab/exam problems, read the descriptions of the problem carefully. Read them twice if you have to. Do not skip anything, or you may find that later questions will not make sense to you. Of course, you should always point out to the instructor/TA if there is anything unclear in the descriptions.

Concerning homework, 4 points must be emphasized: (1) For problem sets involving equations, if appropriate, provide a complete analysis rather than a single number. (2) Be professional in your presentations: write down the unit for your numerical results and round off the final number to 1 or 2 decimal point. If the problem involves a short essay, give it some thoughts and then write it out clearly, precisely, and concisely. (3) You can discuss the problem sets with fellow students, but complete your assignments by yourself. Copying other's work is considered cheating and no points will be given for that homework (even if you only copy one problem out of a total of 6). (4) Hand in the homework on time. Unless otherwise stated, the general timeline is to hand over the homework **in the beginning of the class** a week from when the homework is assigned. Further, if the homework is not handed in on time, for every day it is delayed, 10 points will be taken out of the 100 points assigned to each homework until no points remain.

Exams include (1) multiple quizzes given throughout the semester; (2) midterm; (3) final. The above homework rules (1) and (2) apply. All exams must be handed in at the end of the class. Cheating will incur 0 point for that exam/quiz and a student caught cheating may receive a "F" for the course. A study guide will be provided before the mid and final exams, but not for the quizzes which test the materials you just learned.

Labs discuss several important topics covered in depth. These topics may also appear in lectures. For TA contact info, office hour and other related info, ask TA during Lab one. Unless otherwise mentioned, the lab assignments are expected to be handed in at the end of the lab. For a few big homework assignments, lab time is also used.

Grades: The final grades will be given based on your homework, labs, quizzes and exams. The appropriate percentage is shown:

- Homework 24% (3% × 8 homework)
- Quiz 24% (4% × 6 quizzes)
- Lab 20% (4% × 5 labs)
- Midterm 16%
- Final 16%

Table 1: Letter versus numerical grade

A	B	C	D	F
90-100	80-89	70-79	60-69	< 60

Note that each homework/lab/exam has a stand-alone grade based on 100 points. When determining the final grade, these will be normalized reflecting the percentage distribution above. For example, if you receive the grades below:

Homework:	(1) 80; (2) 0; (3) 95; (4) 70; (5) 80; (6) 90; (7) 75; (8) 89
Quizzes:	(1) 70; (2) 90; (3) 60; (4) 55; (5) 80; (6) 95;
Labs:	(1) 80; (2) 90; (3) 75; (4) 78; (5) 89
Mid term:	80
Final:	90

You final numerical grade out of 100 points will be:

$$
\begin{aligned}
(80+0+95+70+80+90+95+89)\times 3.0\% &+ \\
(70+90+60+55+80+95)\times 4.0\% &+ \\
(80+90+75+78+89)\times 4\% &+ \\
80\times 16\% &+ \\
90\times 16\% & \\
&= 80.01
\end{aligned}
$$

For a final grade of 80.01, you will get a B. The corresponding letter grade is shown in Table 1. Your grade therefore reflects your performance throughout the semester. Since the graduate students (the 5444 group) will have generally better preparation than the undergraduate students (the 4444 group), the final grading will be done separately. The undergraduates will be graded among themselves and the grades scaled accordingly (the grade earned by the top undergraduate will be used as the yardstick to assign grades to the other undergrads). The graduate students will be held at a higher standard and will be graded based on their absolute performance in homework/lab/quiz/exam (rather than scaled among themselves which will not be fair).

Finally, I set high expectation in this class. If you're interested in getting a good grade, be prepared to come to every class, pay full attention, participate in exercises, work out the homework by yourself, hand in homework on time, write professionally (clear, concise, precise, logical), and finally be helpful to your fellow students.

Final thoughts:

The subject of groundwater hydrology is a challenging one though at the same time rewarding. It solves real-world problems using physical principles and mathematical formulations you've been taught ever since grade school. It is rewarding in the sense that your past training can help you understand and solve new problems. Besides, understanding natural processes using physics and mathematics is in itself very interesting. Though you may encounter unfamiliar equations and concepts, keep in mind that your primary goal in this class is to learn something useful rather than getting a grade. Thus, consider this course as a chance to challenge yourself.

Please keep all course materials (notes, exercises, homework, exams, labs) to yourself and do not pass them on to future students. They must, as you have, work to earn the credit.

0.2 Introduction

0.2.1 Groundwater Science

Groundwater hydrology studies the movement of underground water and its dissolved chemical species. It is an importance subject of the applied natural sciences. It emerges from an early engineering root (development of underground water resources) to become, in recent decades, a full-fledged environmental, engineering and geological science. The principles of groundwater hydrology are intimately related to other fields, e.g., petroleum engineering, aqueous geochemistry, soil physics, agricultural engineering, to name just a few, where flow, transport, and reaction through porous media play a fundamental role.

0.2.2 Groundwater Resources

From a practical point of view, securing water of suitable quality is one of the leading environmental concerns of the 21^{st} century. It is estimated that around 80% of diseases and 33% of deaths in the world are related to consumption of contaminated water. With the continued population growth, demands of freshwater supplies are expected to grow, while groundwater is the largest readily-available freshwater reservoir. In many areas of the world (e.g., Nantucket Island, Massachusetts, Saharan desert), it is the only source of freshwater. In the US, it's estimated that about 1/3 of the country is underlain by potable aquifers that can produce at a rate of 50 gallons/minute. In the western US, groundwater is heavily used for irrigation—e.g., the pivotal irrigation system in the High Plains aquifer covering parts of Nebraska, Wyoming, Colorado, Kansas, Oklahoma, Texas and New Mexico.

0.2.3 Groundwater Quality

In the industrialized world, the mid-1900 chemical revolution has introduced petroleum-derived synthetic chemicals into the natural environment. The early disposal of these chemicals was not really regulated. Such chemicals migrate underground, dissolving into groundwater and creating contamination widely (see EPA's website on superfund: http://www.epa.gov/superfund/). The contaminated water often flows away from the source zones, transporting chemicals to wells or other water bodies (lake, stream, reservoirs). These chemicals can be harmful to humans. Famous examples include the Love Canal site, e.g., http://en.wikipedia.org/wiki/Love_Canal, and the Woburn contamination in Massachusetts, e.g., http://www.civil-action.com/.

0.2.4 Groundwater in Wyoming

In Wyoming, groundwater is not only used for domestic purposes, it is also intricately linked to the energy industry. For example, in the Power River Basin, groundwater pumped from the gas-bearing formations contains significant concentrations of dissolved salts; its disposal has become an increasing problem for the economic development of coal bed methane (http://seo.state.wy.us/cbng.aspx). In Laramie, groundwater contamination has also occurred at the Union Pacific Railroad (UPRR) Tie Plant site, see the information on the cleanup efforts in http://deq.state.wy.us/shwd/N_UPRailroad_z03.asp.

0.2.5 What Groundwater Scientists Do

As groundwater scientists, we study a number of issues, roughly divided into these categories (below are excerpts from Fitts, 2002):

1. Water Supply. Water supply wells for drinking water, irrigation, and industrial use. Assemble data on the hydrogeology of the site, e.g., drilling data, well data, and geologic maps. Test wells are drilled and hydraulic testing is used to estimate the aquifer capacity to produce water. Water chemistry is checked to ensure that the water is suitable for its intended use. If all are favorable, a production well is designed and installed.

2. Water Resource Management. To manage aquifers, make decisions such as who is allowed to pump water, how much can be pumped, where wells can be located, where potential contaminant sources to the aquifer are located. The potential impact of surface water projects (location of dams, diversions for irrigation, sewer system emplacement) on groundwater level/quality is also considered.

3. Engineering & Construction. Dewatering: when a deep excavation is made for a building or tunnel, groundwater flows into the pit. Dewatering can reduce the local water table, causing land subsidence. In dam stability analysis, seepage rate and pore water pressure are estimated. Groundwater study is also part of a study on siting landfill and subsurface waste storage locations.

4. Environmental investigation and clean-up. Remediation can involve construction of trenches where contaminants are captured, pump-and-treat, injecting air, chemical solvents, or bacteria, and other schemes.

5. Geologic Processes. Better understand the process involved in the origin of oil, gas, and mineral deposits. Work are also shedding light on past

climates, earthquake generation mechanisms, and geologic hazards (e.g., land slides).

The applications of groundwater science are often interdisciplinary, bridging geology, engineering, environmental sciences, chemistry, biology, and resource management. Problems are typically addressed by a team of people from various disciplines.

0.3 Homework 1

Please hand in your homework in the beginning of the class a week from today.

1. In a few sentences, describe what you already know about groundwater hydrology and what you hope to learn in this class.

2. Among the various fields described in “What Groundwater Scientists Do”, select two categories of your interest. For each category, present a short essay (< 200 words) describing a case study: what kind of problem transpired and what groundwater scientists did to solve the problem. For this, the best way is to do online or library research. The case studies may be from Laramie locally, or, some of the more well known cases in the country, e.g., the Love Canal (see your textbook), the Woburn case upon which the movie Civil Action is based (http://www.geology.ohio-state.edu/courtroom/images.htm).

Please write the essays yourself and provide references as appropriate. Directly copying from somewhere (e.g., someone else’s writing or a report downloaded from the internet) will not do. I will check your writing against such potential sources to make sure there is no plagiarism.

Chapter 1

Basic Math

The following is a review of some basic math you need to know before taking this class (it is in no way complete — more may be added as we proceed in the course). If you find the math incomprehensible, please first take College Algebra and Calculus. **This course is not a math class.** Most equations I will present are derived assuming you know the basic math already.

1.1 Scientific Notation

Scientific Notation is commonly used when dealing with very large and/or very small numbers.

$10 \times 10 = 100 = 10^2 = 1.0 \times 10^2$
$10 \times 10^2 = 10^{1+2} = 10^3 = 1.0 \times 10^3$
$5 \times 10^4 + 60 \times 10^4 = 65 \times 10^4 = 6.5 \times 10^5$
$1 \times 10^3 \times 1 \times 10^4 = 1.0 \times 10^7$
$\frac{1\times10^3}{1\times10^4} = 1 \times 10^{3-4} = 1 \times 10^{-1} = 0.1$
$(1 \times 10^3)^4 = 1.0 \times 10^{12}$
$(1 \times 10^4)^{1/2} = 1.0 \times 10^2$
$0.0000209 = 2.09 \times 10^{-5}$

Note that we often write 1×10^2 as $1.0E2$, or 2.09×10^{-5} as $2.09E{-5}$, etc.

1.2 Decimal Places

In this class, when dealing with non-integer numbers, we adopt (at least) 1 decimal place for the final result, see the example shown in Figure 1.1. During the computation, keep all decimal places. Round the result only in the last step.

We can first write: $V = 5.234 \times 3.72 \times 4.1 = 79.828968 \quad m^3$, then rounding:

$V = 80.0 \quad m^3$ (1 decimal place — required minimum)
$V = 79.83 \quad m^3$ (2 decimal places — preferred)

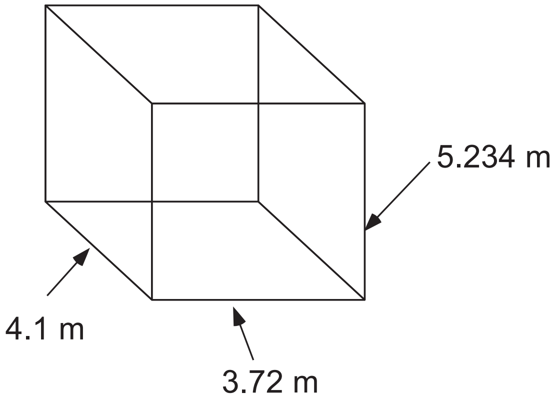

Figure 1.1: What is its volume (V)?

1.3 Dimension Analysis

In the physical sciences, there are 7 Basic *SI Units* (or the metric system — all numbers are related by 10). In this class, we use:

Name	Dimension	Unit
length	[L]	meter(or m)
mass	[M]	kilogram (or kg)
time	[T]	second (or s)

Other units are derived from the basic units:
$Force[F]\ (N) = m\ (kg) \cdot a\ (m/s^2)$
$Pressure\ (Pa) = \frac{Force\ [N]}{Area\ [m^2]} \qquad kg/(m \cdot s^2)$

In Dimension Analysis, the type and dimensions of units at both sides of an equation must be the same. For example, in equation: $Distance\ [L] = Velocity\ [L/T] \times Time\ [T]$, the unit type and dimensions are the same on both sides — $[L]$. The rule applies to more complicated equations as well. This is important for checking the consistency of the equations.

There is also the issue of unit conversion, e.g., between the SI units and other non-standard units, e.g., $1\ m = 3.28084\ feet$, $1\ liter = 0.001\ m^3$, $1\ bar = 10^5 Pa$. Make sure you know how it's done. If you work on a computer with an internet connection, try using:

Online Unit Converter

For many exercises in this class, to ensure that the units are consistent, you need to double-check the units of all relevant quantities before calculation. Sometimes, you can convert the quantities to the SI unit before computation, other times you can work with the English unit. I will usually give you some comments to help you with these exercises. A convenient conversion table is also provided in Appendix

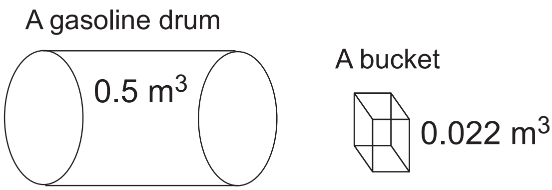

Figure 1.2: What many buckets does it take to fill the drum?

A of your textbook (Fitts, 2002).

In doing the exercise of this study, there is yet another type of "unit" you must comprehend. In Figure 1.2, do you know what unit you should use when presenting the result?

The answer is: $\frac{0.5}{0.022} \frac{m^3}{(m^3/bucket)} = 22.7 \qquad (buckets)$.

1.4 Logarithm & Exponential

The logarithm of a number is the exponent to which a base number must be raised to yield the value of the number. There are two common bases: 10 and e=2.718... (natural logarithm).

$10^2 = 100; \qquad log_{10}100 = 2$ or $log100 = 2$
$10^{-2} = 0.01; \qquad log0.01 = -2$
$10^0 = 1; \qquad log1 = 0$
$e^0 = 1; \qquad ln1 = 0$
$25^{1/2} = 5; \qquad log_{25}^5 = 1/2$

If a, b are constants:
$a^x a^y = a^{x+y}$
$a^0 = 1$ (if $a \neq 0$)
$(ab)^x = a^x b^x$
$(a^x)^y = a^{xy}$
$a^{-x} = 1/a^x$
$a^x/a^y = a^{x-y}$

Logarithms have the following relationships:
$b = e^a \Longleftrightarrow lnb = a$
$b = 10^a \Longleftrightarrow logb = a$
$lne = 1$
$ln1 = log1 = 0$
$log10 = 1$
$ln(ab) = lna + lnb$ (same applied to "$log(ab)$")
$ln(a/b) = lna - lnb$
$ln(a^b) = blna$
$ln(1/b) = -lnb$
$ln(a) \simeq 2.30log_{10}a$

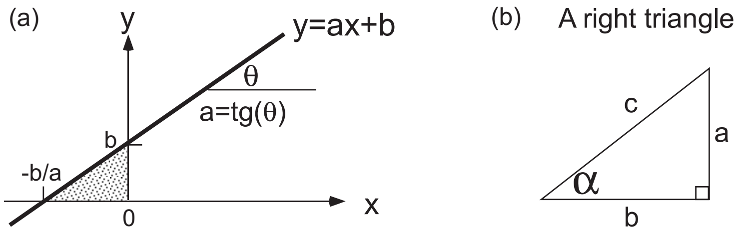

Figure 1.3: A line function (a) and a right triangle (b).

Exponential functions are defined as $f(x) = a^x + B$, where a is a real constant and B is any expression. Commonly, exponential function is e-based, for example, $f(x) = e^{-x}$ is an exponential function ($a = e$), and, $\ln f(x) = -x$.

1.5 Areas, Volumes, Circumferences

Be familiar with the calculation of different areas (A) and volumes.

Area of a circle with radius r: $A = \pi r^2$, $pi = 3.14159265\ldots$

Circumference of a circle with a diameter d: πd

Area of a triangle with a base b and altitude h: $A = (1/2)bh$

Area of a parallelogram with sides a and b and an included angle θ: $A = a \cdot b \cdot sin\theta$.

Volume of a cylinder with radius r and height h: $V = \pi r^2 h$.

1.6 Analytic Geometry & Trigonometric Functions

Equation of a straight line in rectilinear coordinates (Figure 1.3a):

$y = ax + b$

a is the slope, b is the intercept of the y axis.

In Figure 1.3b, for a right triangle, we have:

$c^2 = a^2 + b^2; \qquad c = \sqrt{a^2 + b^2}$

Common trigonometric functions:

$sin\alpha = a/c$

$cos\alpha = b/c$

$tg\alpha = a/b$

1.7 Integration

Most analytic solutions for hydrologic problems are developed based on integration and differentiation. You should know the basics from your calculus class. For example, to find the area beneath a function (Figure 1.4), we can use:

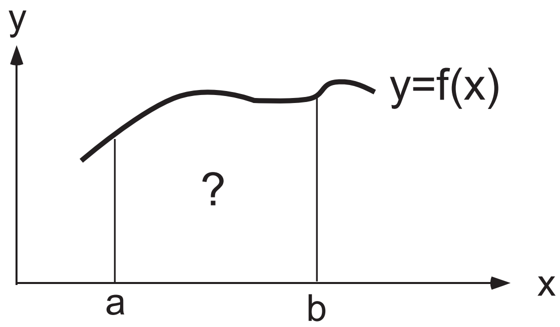

Figure 1.4: What is the area underneath f(x), between a and b?

$Area = \int_a^b f(x)dx$

What is the area of the grey right triangle is Figure 1.3a?

$Area = \int_{-b/a}^{0}(ax+b)\ dx = (\frac{a}{2}x^2+bx)|_{-b/a}^{0} =$
$(\frac{a}{2}x^2+bx)|_{x=0} - (\frac{a}{2}x^2+bx)|_{x=-b/a} =$
$0 - (\frac{a}{2}\frac{b^2}{a^2} + b(-b/a)) = -(\frac{b^2}{2a} - \frac{b^2}{a}) = \frac{b^2}{2a}$

From the area relation for right triangle, we can directly calculate: $(b/a) \times b/2 = b^2/2a$

Some simple rules of integration (all rigorously derivable!): x is an independent variable, c, a, b are expressions not involving x (e.g., constant), n is an integer.

Indefinite integral:
$\int 1dx = x$
$\int dx = x$
$\int d[f(x)] = f(x)$

Definite integral:
$\int_a^b dx = x|_a^b = b - a$
$\int_a^b cdx = cx|_a^b = c(b-a)$
$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
$\int_a^b x^n dx = \frac{x^{n+1}}{n+1}|_a^b = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}$
$\int_a^b (1/x)dx = \ln x|_a^b = lnb - lna = ln(b/a)$

1.8 Slope

For a straight line: $y = \tan(\alpha)x + b$ (Figure 1.5), the slope is defined by:

slope=$\tan(\alpha) = \frac{y(x+\Delta x)-y(x)}{\Delta x}$

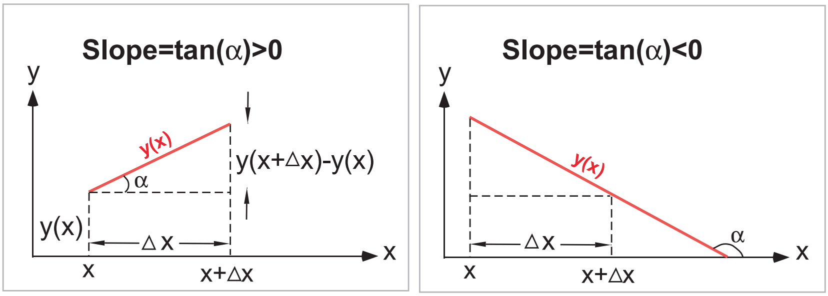

Figure 1.5: A straight line function. The slope of the line is $\tan(\alpha)$.

Clearly, depending on whether y is increasing with x or not, the slope can be positive or negative.

1.9 Differentiation

Many equations and formulations of this class are given in differentials. By definition,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x}$$

Do you notice the similarity to the definition of the "slope"? Just like slope, dy/dx **can be a positive or negative value.** Since the function may be curved, the term "gradient" takes the place. For example, $y = x^2 + 1$, by definition (above), its gradient is $dy/dx = d/dx(x^2 + 1) = 2x$. So, its gradient is a function which varies with x (try plotting both $y(x)$ and dy/dx out).

The **slope** is just a special form of gradient when the function is a straight line, e.g., $y(x) = ax+b$. By the definition of gradient, $dy/dx = d/dx(ax+b) = a$. So, when the function is a straight line, the gradient of this function is a constant (see the tangent of the angle in Figure 1.5 which remains unchanged along x).

During this class, we often write the differential in terms of a difference (or a gradient):

$$\frac{dy}{dx} \simeq \frac{\Delta y}{\Delta x} = \frac{y(x + \Delta x) - y(x)}{\Delta x}$$

More importantly, depending on what the function looks like and where the coordinate x points to, $\frac{\Delta y}{\Delta x}$ may be positive or negative, just like the slope we analyzed above. Make sure you know this above difference by heart (memorize it!).

A good tutorial to refresh your memory:
http://www.mathsyear2000.org/alevel/pure/purtutdifint.htm

Gradient of the hydraulic head

In this course, we use h to represent the hydraulic head (more on this later). If the hydraulic head is a 1D function of x: $h = h(x)$, its gradient is:

$$\frac{dh}{dx} \simeq \frac{\Delta h}{\Delta x} = \frac{h(x+\Delta x) - h(x)}{\Delta x}$$

If the hydraulic head is a 3D function: $h = h(x, y, z)$, its gradient becomes:

$$\nabla h = \left\{ \begin{array}{c} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial z} \end{array} \right\}$$

Note that now the head gradient is a vector with 3 components. Each component is a partial derivative of h with respect to a particular coordinate axis. For example, by definition:

$$\frac{\partial h}{\partial x} = \lim_{\Delta x \to 0} \frac{h(x+\Delta x, y, z) - h(x, y, z)}{\Delta x}$$

This derivative thus evaluates the head gradient along the x direction, keeping y and z fixed. We can similarly write the definitions out for $\frac{\partial h}{\partial y}$ and $\frac{\partial h}{\partial z}$.

If we only evaluate h along a 2D vertical cross section (x,z), the hydraulic gradient has only two components:

$$\nabla h = \left\{ \begin{array}{c} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial z} \end{array} \right\}$$

where the partial derivatives:

$$\frac{\partial h}{\partial x} \simeq \frac{h(x+\Delta x, z) - h(x, z)}{\Delta x}$$
$$\frac{\partial h}{\partial z} \simeq \frac{h(x, z+\Delta z) - h(x, z)}{\Delta z}$$

Where $\frac{\partial h}{\partial x}$ is evaluated by looking at how head varies along x, keeping z fixed. $\frac{\partial h}{\partial z}$ is evaluated by looking at how head varies along z, keeping x fixed. In Chp4, Exercise 5, we'll use the above formulation for a 2D flow analysis. When we do that exercise, we'll come back to the gradient definition presented here. You will see why we chose those particular locations to calculate the particular $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial z}$ for that exercise.[1]

What if our head is only varying along the 2D planeview coordinate (x,y)? How would you write the hydraulic gradient vector and its components?

$$\nabla h = \left\{ \begin{array}{c} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \end{array} \right\}$$

[1] For example, to evaluate $\frac{\partial h}{\partial x} \simeq \frac{h(x+\Delta x, z) - h(x,z)}{\Delta x}$, we choose two points in the aquifer: $P1$ and $P2$. If point $P1$ lies at the same elevation (z) as point $P2$, but $P1$ occurs at a **higher** value along the x axis than $P2$ (this will depend on where the x axis is pointing), we can set the horizontal distance from $P2$ to $P1$ as Δx, thus $h(x + \Delta x) = h(P1)$, $h(x) = h(P2)$, then the partial derivative is evaluated as: $\frac{\partial h}{\partial x} \simeq \frac{h(P1) - h(P2)}{\Delta x}$.

where the partial derivatives are, by definition:

$$\frac{\partial h}{\partial x} \simeq \frac{h(x+\Delta x, y) - h(x,y)}{\Delta x}$$
$$\frac{\partial h}{\partial y} \simeq \frac{h(x, y+\Delta y) - h(x,y)}{\Delta y}$$

Their meanings and how we'll calculate them for a particular coordinate system are similar to above.

Some simple rule of differentiation: (a is a constant)
$\frac{da}{dx} = 0$
$\frac{dx}{dx} = 1$
$\frac{d(ax)}{dx} = \frac{adx}{dx} = a$
$\frac{d(x^n)}{dx} = nx^{n-l}$
e.g., $\frac{d(x^2)}{dx} = 2x$, $\frac{d(1/x)}{dx} = -(1/x^2)$
$\frac{d(lnx)}{dx} = 1/x$
$\frac{d(e^x)}{dx} = e^x$
$\frac{d(a^x)}{dx} = a^x \ln a$

u, v are independent variables:
$d(u+v) = du + dv$
$d(uv) = udv + vdu$
$d(u/v) = (vdu - udv)/v^2$

If $f(x)$ and $g(x)$ are a function of x:
$\frac{d}{dx}[af(x)] = a\frac{df(x)}{dx}$
$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{df(x)}{dx} + f(x)\frac{dg(x)}{dx}$

Commonly, chain-rule is applied:
$\frac{d}{dx}g(f(x)) = \frac{dg}{df}\frac{df}{dx}$
e.g., $\frac{d(e^{ax})}{dx} = e^{ax}\frac{d(ax)}{dx} = ae^{ax}$
$\frac{d(e^{x^3})}{dx} = e^{x^3}\frac{d(x^3)}{dx} = e^{x^3}(3x^2)$

Exercise 1 *Calculate the weight of freshwater in a cylindrical tank that has a diameter of 4 ft and a height of 6 ft. (Note, convert everything to the SI unit.)*

Exercise 2 *Calculate the volume of water in an aquifer shown in Figure 1.6 given a porosity of 0.1. This aquifer representation is often used in this class: we see that the actual 3D geometry is represented by a vertical cross section in the x-z plane and a unit aquifer thickness in the y direction.*

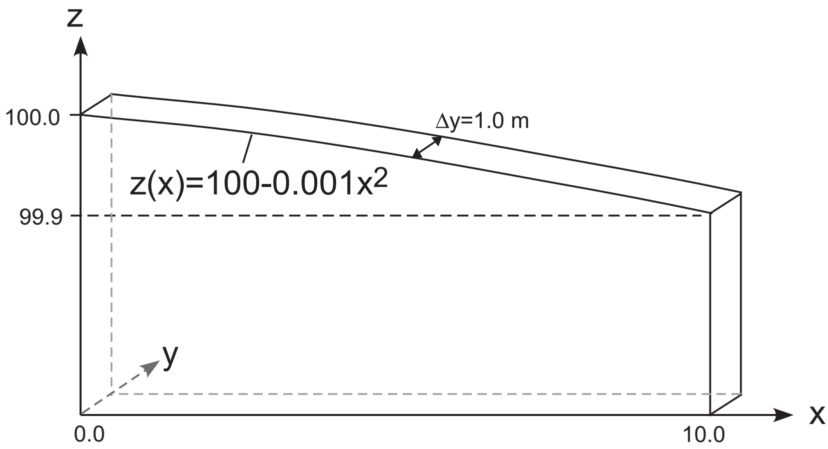

Figure 1.6: A vertical slice of an aquifer.

1.10 Functions of One Variable

An example function of the one variable x is: $h(x) = x^2 + 2x - 1$

The first derivative of this function is: $dh/dx = 2x + 2$

The second derivative of this function is: $\frac{d^2h}{dx^2} = \frac{d}{dx}(\frac{dh}{dx}) = 2$

In mathematics, the function h is also called a *dependent* variable, while the spatial axis x called an *independent* variable. Some physical sense can be made out of this terminology. For example, if the hydraulic head is varying along a x-y plane: $h = h(x, y)$ (draw a regional confined aquifer and its potentiometric surface on the board to help students visualize that), head is changing as x and y changes, so head depends on x and y. HOWEVER, the spatial axes themselves can change irrespective of what head (or any other physical quantity) is. AND, x changes irrespective of y, vice versa. In a more general case, head is varying in 3D and over time, so we have $h = h(x, y, z, t)$, in this case, head depends on x, y, z, t, though each of these are independent of head and independent of one another. These are thus the independent variables.

An ordinary differential equation (ODE) is an equation containing derivatives of a function of one variable. For example, $h(x) = x^2 + 2x - 1$ is a solution of the following ODE: $\frac{d^2h}{dx^2} = \frac{dh}{dx} - 2x$ (this can be verified by substituting the pervious derivatives into the RHS and LHS of this ODE).

Some simple hydrological systems can be described along 1D axis and steady state (it means the head does not change with time, more on this later), in which case we typically solve a ODE to find, e.g., $h(x)$. This solution can use analytical methods (a sperate class in Math department exists in just solving ODE) or numerical methods. Such problems will be the topics of an advanced groundwater modeling class I'll teach in future.

1.11 Functions of two or more variables

If you differentiate a function of two or more variables with respect to one of the variables, you have a partial derivative. Consider the following function (Fitts, 2002):

$h = h(x, y) = 4x^2 + 3y + 10xy^2$

The partial derivative with respect to x is evaluated just like the derivative with respect to x, treating y as though it were a constant. The function h has the following partial derivatives:

$\frac{\partial h}{\partial x} = 8x + 10y^2$

$\frac{\partial^2 h}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial h}{\partial x} = \frac{\partial}{\partial x}(8x + 10y^2) = 8$

$\frac{\partial h}{\partial y} = 3 + 20xy$

$\frac{\partial^2 h}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial h}{\partial y} = \frac{\partial}{\partial y}(3 + 20xy) = 20x$

$\frac{\partial}{\partial x}\frac{\partial h}{\partial y} = \frac{\partial}{\partial x}(3 + 20xy) = 20y$

$\frac{\partial}{\partial y}\frac{\partial h}{\partial x} = \frac{\partial}{\partial y}(8x + 10y^2) = 20y$

The example function h is a solution of the following partial differential equation (PDE): $\frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial x^2} = 20x + 8$.

The idea of a dependent and independent variables holds here too. In hydrology, for more realistic flow configurations, we are generally interested in solving the properties (e.g., head, velocity, solute concentration) in higher dimensions and evaluating their changes over time, e.g., $h = h(x, y, z, t)$. In these cases, we typically solve a PDE using either analytical means (e.g., turning PDE to ODE, Fourier Transform, and some other means), or numerical means. The numerical methods of solving such PDE will be described in the modeling class.

1.12 Scalar, Vector, Matrix

Scalar is a mathematical quantity that has only 1 component, e.g., density, pressure, temperature. Vector is a quantity that has 3 components in three-dimensional coordinate and 2 components in two-dimensional coordinate. Vector reduces to a scalar in one-dimensional coordinate.

A typical example of a vector is velocity of a canon propelled through space (Fig 1.7). If we choose to analyze its trajectory in two-dimensions, the velocity magnitude ($|v| = \sqrt{v_x^2 + v_z^2}$) of the ball remains constant through time, but its direction is changing. Thus, the two component velocities (v_x, v_z) are changing in time. Notice the difference between $\vec{v_A}$ and $\vec{v_C}$: why would the vertical component of $\vec{v_C}$ be negative? Answer: whether this value is positive or negative depends on the adopted coordinate system: the z axis points upward, but the vertical component of $\vec{v_C}$ points in the opposite direction (along -z axis), that's way it is negative. Now, what happens when we rotate the axis so x faces the opposite direction? Answer: The sign of the component v_x changes; but the component v_z does not change and the velocity magnitude does not change. So, in the new coordinate, $\vec{v_A} = [-1.0, 1.5]$, $\vec{v_B} = [-1.8, 0.0]$, $\vec{v_C} = [-1.0, -1.5]$. **Make sure you thoroughly understand this convention—throughout**

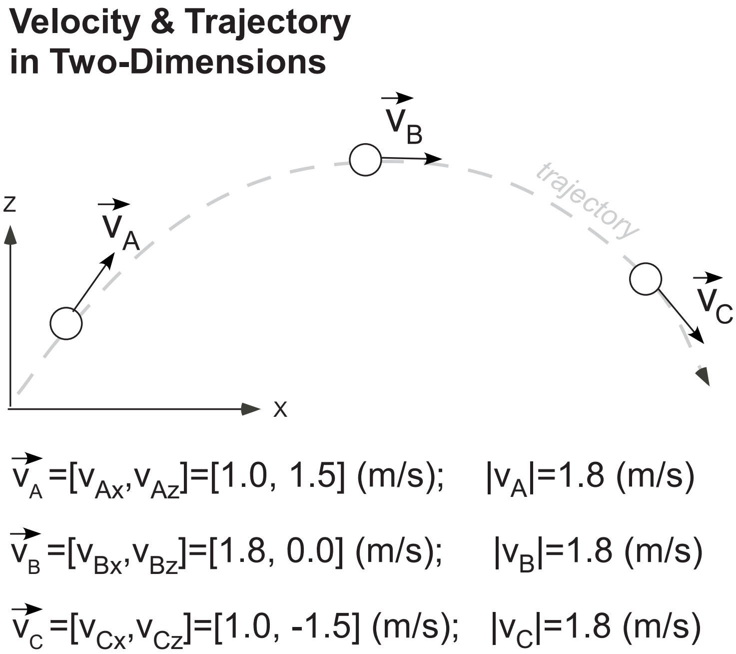

Figure 1.7: The trajectory and velocity of a ball at three locations A, B, C.

this course, we are constantly adding or dropping the negative sign in response to the coordinate we use!

In 2D or 3D, the relationship between a vector and its components (which is just the normal projection of the vector onto each coordinate axis) is shown in Figure 1.8. Make sure you understand this.

An important property of vectors is the vector *inner product* or *dot product*. If $\vec{a} = \{a_1, a_2, \ldots, a_n\}$, $\vec{b} = \{b_1, b_2, \ldots, b_n\}$, their inner product is:

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$$

Note that the result of two vector inner project is a scalar quantity.

In hydrology (as well as many other physical sciences), an important vector is the gradient vector ∇:

$$\nabla = \left\{ \begin{array}{c} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{array} \right\}$$

The gradient vector is consider a mathematical operator. Two operations are of particular relevance in this class: (1) ∇ operating on a scalar gives a *vector*,

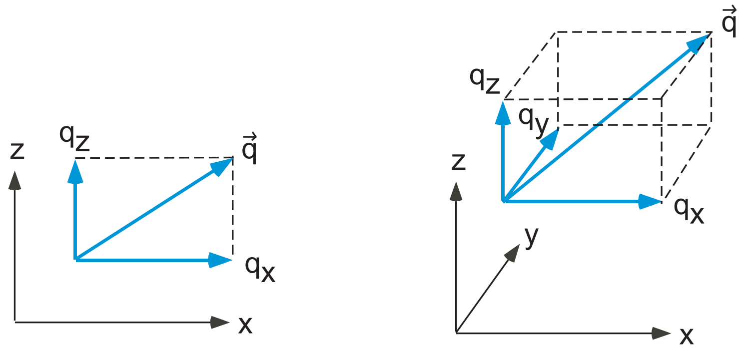

Figure 1.8: Relation between a vector ($\vec{q}$) and its components in 2D and 3D.

e.g., the previous hydraulic gradient:

$$\nabla h = \left\{ \begin{array}{c} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial z} \end{array} \right\}$$

(2) vector product between ∇ and another vector gives a *scalar*. For example,the velocity vector in 3D has 3 components: $\vec{q} = \{q_x, q_y, q_z\}$, the vector product between ∇ and $\vec{q}$ gives:

$$\nabla \cdot \vec{q} = \left\{ \begin{array}{c} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{array} \right\} \cdot \left\{ \begin{array}{c} q_x \\ q_y \\ q_z \end{array} \right\} = \frac{\partial}{\partial x}(q_x) + \frac{\partial}{\partial y}(q_y) + \frac{\partial}{\partial z}(q_z)$$

Matrix

Now, what is matrix? Within the context of the linear algebra, it is a two-dimensional (2D) data structure. A square $n \times n$ matrix can generally be written as:

$$A_{n\times n} = \left[\begin{array}{cccc} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{array} \right]$$

In linear algebra, there is an important relationship on matrix-vector multiplication (since we won't have time to review the subject of Linear Algebra, you should know this relationship by heart):

$$\left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right] \left\{ \begin{array}{c} b_1 \\ b_2 \end{array} \right\} = \left\{ \begin{array}{c} a_{11}b_1 + a_{12}b_2 \\ a_{21}b_1 + a_{22}b_2 \end{array} \right\} \tag{1.1}$$

We see that this multiplication outcome is a vector with 2 components. The above relation can be extended to higher dimensions, e.g., we can write the vector outcome for a multiplication of a 3×3 matrix and 3× 1 vector:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} = \begin{Bmatrix} a_{11}b_1 + a_{12}b_2 + a_{13}b_3 \\ a_{21}b_1 + a_{22}b_2 + a_{23}b_3 \\ a_{31}b_1 + a_{32}b_2 + a_{33}b_3 \end{Bmatrix}$$

The resulting vector has 3 components or a 3× 1 vector. This can be extended for the multiplication of n×n matrix and n×1 vector.

1.13 The Summation Sign

We use the summation sign to condense the writing of long equations, for example:

$$x_1 + x_2 + \ldots + x_{10} = \sum_{i=1}^{10} x_i$$

$$y_1 + y_2 + \ldots + y_{m-1} + y_m = \sum_{j=1}^{m} y_j$$

$$(x_1 - a)^2 + (x_2 - a)^2 + \ldots + (x_n - a)^2 = \sum_{k=1}^{n} (x_k - a)^2$$

In these equations, i, j, k are called summation indexes. These are flexible symbols decided by us to use, e.g., we can use p, q, o as indexes instead. Later on, we may briefly explain the convention of "Einstein Summation" which is sometimes used to condense long equations.

A constant can often be taken in and out of a summation sign, for example, below is a simple proof:

$$\sum_{i=1}^{10} ax_i = ax_1 + ax_2 + \ldots + ax_{10} = a(x_1 + x_2 + \ldots + x_{10}) = a \sum_{i=1}^{10} x_i$$

Thus, we can write:

$$\sum_{i=1}^{m} b(x_i - c)^3 = b \sum_{i=1}^{m} (x_i - c)^3$$

1.14 Test 1

This quiz will be given in the next class after Chapter 1 is taught.