# ECON 5110 Solutions to Problem Set #1

1. Romer 3rd edition, Exercise 1.5.

   Answer:

   (a) The equation describing the evolution of the capital stock per unit of effective labor is given by

   $$\dot{k} = sk^{\alpha} - (n + g + \delta)k.$$

   On the balanced-growth path, $\dot{k} = 0$. Denoting the balanced-growth-path value of $k$ as $k^*$, we have

   $$s(k^*)^{\alpha} = (n + g + \delta)k^*.$$

   Rearranging to solve for $k^*$ yields

   $$k^* = [s/(n + g + \delta)]^{1/(1-\alpha)}.$$

   Output per unit of effective labor along the balanced-growth path is

   $$y^* = [s/(n + g + \delta)]^{\alpha/(1-\alpha)}.$$

   Consumption per unit of effective labor can be found from $c^* = (1 - s)y^*$

   $$c^* = (1 - s)[s/(n + g + \delta)]^{\alpha/(1-\alpha)}.$$

   (b) Using the balanced-growth condition to solve for $s$ gives

   $$s = (n + g + \delta)(k^*)^{(1-\alpha)}.$$

   Substituting this expression into $c^*$ from part (a) with some light algebra produces

   $$c^* = (k^*)^{\alpha} - (n + g + \delta)k^*.$$

   To find the maximum sustainable consumption per effective worker along the balanced growth path, take the derivative with respect to $k$ and set equal to zero:

   $$\frac{\partial c^*}{\partial k^*} = \alpha(k^*)^{(\alpha-1)} - (n + g + \delta) = 0.$$

This condition implies that $f'(k^*)$ must equal $(n+g+\delta)$ to be at the golden-rule level of consumption per effective worker. The associated level of capital per effective worker is

$$k^*_{GR} = [\alpha/(n+g+\delta)]^{1/(1-\alpha)}.$$

(c) Substituting $k^*_{GR}$ into the expression for $s$ in part (b) gives

$$s_{GR} = (n+g+\delta)[\alpha/(n+g+\delta)]^{(1-\alpha)/(1-\alpha)} = \alpha.$$

2. Romer 3rd edition, Exercise 1.15.

Answer:

Taking logs of the production function gives

$$\ln Y(t) = \alpha \ln K(t) + \beta \ln R(t) + \gamma \ln T(t) + (1-\alpha-\beta-\gamma)[\ln A(t) + \ln L(t)].$$

Differentiating with respect to time gives the growth rates of the variables:

$$g_Y(t) = \alpha g_K(t) + \beta g_R(t) + \gamma g_T(t) + (1-\alpha-\beta-\gamma)[g_A(t) + g_L(t)].$$

Substituting in the growth rates for $T$, $R$, $A$ and $L$ gives

$$g_Y(t) = \alpha g_K(t) + (\beta+\gamma)n + (1-\alpha-\beta-\gamma)(g+n).$$

We know that $Y$ and $K$ must grow at equal rates on the balanced growth path because

$$\frac{\dot{K}(t)}{K(t)} = s\frac{Y(t)}{K(t)} - \delta.$$

Therefore, we have

$$g_Y(t) = \frac{(\beta+\gamma)n + (1-\alpha-\beta-\gamma)(g+n)}{(1-\alpha)}.$$

Finally, equation (1.50) is

$$g_{Y/L}(t) = g_Y(t) - g_L(t) = \frac{(\beta+\gamma)n + (1-\alpha-\beta-\gamma)(g+n)}{(1-\alpha)} - n = \frac{(1-\alpha-\beta-\gamma)g}{(1-\alpha)}.$$

3. Romer 2nd edition, Exercise 2.2.

Answer:

(a) Utility, with the budget constraint substituted in, is

$$U(C_1) = \frac{C_1^{1-\theta}}{1-\theta} + \beta \frac{[(W - P_1 C_1)/P_2]^{1-\theta}}{1-\theta}$$

where $\beta = 1/(1+\rho)$. The first-order condition is

$$\frac{dU(C_1)}{dC_1} = C_1^{-\theta} - \beta C_2^{-\theta} \frac{P_1}{P_2} = 0.$$

Rearranging gives

$$\frac{C_1}{C_2} = \left(\beta \frac{P_1}{P_2}\right)^{-\frac{1}{\theta}}.$$

Substituting in the budget constraint and rearranging produces

$$\begin{aligned} C_1 &= \frac{W}{P_2}\left[1 - R^{\frac{\theta-1}{\theta}} \beta^{-\frac{1}{\theta}}\right]^{-1} \left[(R\beta)^{-\frac{1}{\theta}}\right] \\ C_2 &= \frac{W}{P_2}\left[1 - R^{\frac{\theta-1}{\theta}} \beta^{-\frac{1}{\theta}}\right]^{-1} \left[1 - 2R^{\frac{\theta-1}{\theta}} \beta^{-\frac{1}{\theta}}\right] \end{aligned}$$

where $R = P_1/P_2$.

(b) Taking logs of the ratio in part (a) gives

$$\ln(\frac{C_1}{C_2}) = \alpha - \frac{1}{\theta}\ln(\frac{P_1}{P_2})$$

where $\alpha = -\frac{1}{\theta}\ln(\beta)$. Therefore, the elasticity of substitution between $C_1$ and $C_2$ is

$$-\frac{\partial \ln(C_1/C_2)}{\partial \ln(P_1/P_2)} = \frac{1}{\theta}.$$

4. Romer 2nd edition, Exercise 2.6.

Answer:

(a) The $\dot{k} = 0$ curve is given by

$$c(t) = f(k(t)) - (n+g)k(t). \quad (1)$$

Therefore, a permanent fall in $g$ will cause the $\dot{k} = 0$ curve to shift up. The value of $k$ that is associated with $c = 0$ will increase.

(b) The $\dot{c} = 0$ curve is given by

$$f'(k(t)) = \rho + \theta g.$$

Therefore, a permanent fall in $g$ will cause the $\dot{c} = 0$ curve to shift to the right.

(c) The new steady-state values of $k$ and $c$ will be higher. However, before reaching its new steady state, the economy must jump onto its new saddle path. Since the value of $k$ is predetermined, it cannot jump discontinuously. As a result, the value of $c$ will jump immediately to make sure the economy is on its new saddle path.

(d) Denoting $k_*$ as the value of $k$ on the balanced-growth path and $s = [f(k) - c]/f(k)$ as the fraction of output that is saved, we can use the equation in part (a) to derive

$$\frac{\partial s(t)}{\partial g} = \frac{1}{f(k_*)}\left\{k_* + \frac{\partial k_*}{\partial g}(n+g)\left[1 - \frac{k_* f'(k_*)}{f(k_*)}\right]\right\}.$$

From part (c), we know that $\frac{\partial k_*}{\partial g} < 0$. However, this is not sufficient to sign the derivative because we do not know whether $k_* f'(k_*)/f(k_*)$ is greater than or less than one.

(e) Using the functional form, $f(k) = k^\alpha$, we get

$$k_* = \left[\frac{\rho + \theta g}{\alpha}\right]^{\frac{1}{\alpha - 1}}$$

and

$$\begin{aligned}\frac{\partial s(t)}{\partial g} &= k_*^{-\alpha}\left\{k_* + \frac{\partial k_*}{\partial g}(n+g)\left[1 - \alpha\frac{k_* k_*^{\alpha-1}}{k_*^\alpha}\right]\right\} \\ &= k_*^{1-\alpha} + k_*^{-\alpha}\frac{\partial k_*}{\partial g}(n+g)(1-\alpha)\end{aligned}$$

where

$$\frac{\partial k_*}{\partial g} = \frac{\theta}{\alpha(\alpha - 1)}\left[\frac{\rho + \theta g}{\alpha}\right]^{\frac{2-\alpha}{\alpha-1}} < 0.$$

With the appropriate substitutions, we get

$$\frac{\partial s(t)}{\partial g} = \left(\frac{\alpha}{\rho + \theta g}\right)\left(1 - \frac{\theta(n+g)}{\rho + \theta g}\right).$$

This expression is positive if

$$\rho + \theta g > \theta(n+g).$$

Assuming $\theta \leq 1$, this is true because the marginal product of capital at the *modified* golden-rule level of capital (left-hand side) is greater than the marginal product of capital at the *absolute* golden-rule level of capital (right-hand side divided by $\theta$). Therefore, if $\theta \leq 1$ and the production function is Cobb-Douglas, a fall in productivity will lead to a fall in the savings rate. If $\theta > 1$, we cannot sign $\partial k_*/\partial g$.