ECON 5110 Solutions to Problem Set #2

1. Solving an RBC model.

   Answer: Begin by maximizing the social planner's objective function with the constraints substituted in

$$\begin{aligned}\max_{\{k_{t+1},n_t\}_{t=0}^{\infty}} V_t &= (1-\phi)\log[-k_{t+1}+(1-\delta)k_t+z_t k_t^{1-\theta}n_t^{\theta}]+\phi\log[1-n_t]\\ &+\beta E_t\left\{(1-\phi)\log[-k_{t+2}+(1-\delta)k_{t+1}+z_{t+1}k_{t+1}^{1-\theta}n_{t+1}^{\theta}]+\phi\log[1-n_{t+1}]\right\}+\cdots\end{aligned}$$

   The first-order conditions are

$$\begin{aligned}\frac{\partial V_t}{\partial k_{t+1}} &= -\frac{(1-\phi)}{c_t}+\beta E_t\left\{\frac{(1-\phi)}{c_{t+1}}((1-\delta)+(1-\theta)\frac{y_{t+1}}{k_{t+1}})\right\}=0\\ &\Longrightarrow \frac{1}{c_t}=\beta E_t\left\{\frac{1}{c_{t+1}}((1-\delta)+(1-\theta)\frac{y_{t+1}}{k_{t+1}})\right\} \qquad (1)\end{aligned}$$

   and

$$\begin{aligned}\frac{\partial V_t}{\partial n_t} &= \frac{(1-\phi)}{c_t}\theta\frac{y_t}{n_t}-\frac{\phi}{1-n_t}=0\\ &\Longrightarrow \frac{(1-\phi)}{c_t}\theta\frac{y_t}{n_t}=\frac{\phi}{1-n_t}. \qquad (2)\end{aligned}$$

   Together with z_0, k_0, the resource constraint

$$c_t+k_{t+1}-(1-\delta)k_t=z_t k_t^{1-\theta}n_t^{\theta} \qquad (3)$$

   and the technology shock process

$$z_t=z_{t-1}^{\rho}\exp(\epsilon_t), \qquad (4)$$

   we have the entire system.

   (a) Steady state. The steady-state equations are

$$\begin{aligned}1 &= \beta((1-\delta)+(1-\theta)\frac{y}{k})\\ \frac{(1-\phi)}{c}\theta\frac{y}{n} &= \frac{\phi}{1-n}\\ \frac{c}{k}+\delta &= k^{-\theta}n^{\theta}.\end{aligned}$$

The steady-state equations can be re-organized to get

$$\begin{aligned}
\frac{y}{k} &= \frac{1}{(1-\theta)}(\frac{1}{\beta}-(1-\delta));\ \frac{c}{k}=\frac{y}{k}-\delta;\\
\frac{c}{y} &= \frac{c}{k}\frac{k}{y};\ n=\left[1+\frac{\phi}{(1-\phi)\theta}\frac{c}{y}\right]^{-1}.
\end{aligned}$$

(b) Linearization. The linearized equations are

$$\begin{aligned}
\hat{c}_t &= E_t\hat{c}_{t+1}-[\beta(1-\theta)\frac{y}{k}]E_t\hat{y}_{t+1}+[\beta(1-\theta)\frac{y}{k}]E_t\hat{k}_{t+1} && (5)\\
\hat{y}_t &= \hat{c}_t+[(1-n)^{-1}]\hat{n}_t && (6)\\
[\frac{c}{y}]\hat{c}_t+[\frac{k}{y}]\hat{k}_{t+1} &= \hat{z}_t+[(1-\theta)+(1-\delta)\frac{k}{y}]\hat{k}_t+\theta\hat{n}_t && (7)\\
\hat{z}_t &= \rho\hat{z}_{t-1}+\epsilon_t. && (8)
\end{aligned}$$

Next, simplify to a three-variable system by substitution to get

$$\begin{aligned}
a_4\hat{c}_t+a_5\hat{k}_t &= [\frac{k}{y}]\hat{k}_{t+1}-a_3\hat{z}_t\\
\hat{c}_t &= a_6E_t\hat{c}_{t+1}+a_7E_t\hat{k}_{t+1}+a_8E_t\hat{z}_{t+1}
\end{aligned}$$

and equation (8), where

$$\begin{aligned}
a_1 &= \frac{(1-n)\theta}{(1-n)\theta-1};\ a_2=\frac{-(1-\theta)}{(1-n)\theta-1};\ a_3=a_2/(1-\theta);\\
a_4 &= a_1-\frac{c}{y};\ a_5=a_2+(1-\delta)\frac{k}{y};\ a_6=(1-\beta(1-\theta)\frac{y}{k}a_1);\\
a_7 &= \beta(1-\theta)\frac{y}{k}(1-a_2);\ a_8=-\beta(1-\theta)\frac{y}{k}a_3.
\end{aligned}$$

(c) Matrix Form. The system can be written as

$$F\Gamma_t=G\Gamma_{t+1}+Hv_{t+1}$$

where

$$\begin{aligned}
F &= \begin{bmatrix} a_5 & a_4 & a_3\\ 0 & 1 & 0\\ 0 & 0 & \rho \end{bmatrix};\ G=\begin{bmatrix} k/y & 0 & 0\\ a_7 & a_6 & a_8\\ 0 & 0 & 1 \end{bmatrix};\ H=\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & a_7 & a_6 & a_8\\ -1 & 0 & 0 & 0 \end{bmatrix};\\
\Gamma_t &= (\hat{k}_t,\hat{c}_t,\hat{z}_t)';\ v_{t+1}=(\epsilon_{t+1},w^k_{t+1},w^c_{t+1},w^z_{t+1})';\ w^x_{t+1}=E_tx_{t+1}+x_{t+1}.
\end{aligned}$$

(d) Diagonalization. Next, we diagonalize the A matrix from

$$\Gamma_t = A\Gamma_{t+1} + F^{-1}Hv_{t+1} \tag{9}$$

where $A = F^{-1}G$. The matrix A can be written as $A = Q\Lambda Q^{-1}$, where Λ is a diagonal matrix with the three eigenvalues along the main diagonal and Q is a matrix of stacked eigenvectors. Substituting $A = Q\Lambda Q^{-1}$ into (9), iterating forward and taking expectations conditional on time t information gives

$$q^{11}\hat{k}_t + q^{12}\hat{c}_t + q^{13}\hat{z}_t = 0,$$

where $q^{(1)} = (q^{11}, q^{12}, q^{13})$ is the first row of Q^{-1}. This assumes that the forward stable eigenvalue is in the $[1,1]$ position of Λ.

(e) VAR for the state vector. With substitutions, we can write the state equations as

$$\begin{bmatrix} \hat{k}_{t+1} \\ \hat{z}_{t+1} \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ 0 & \rho \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{z}_t \end{bmatrix} + \begin{bmatrix} 0 \\ \epsilon_{t+1} \end{bmatrix}$$

where

$$b_1 = [\frac{y}{k}(a_5 - a_4)\frac{q^{11}}{q^{12}}] \text{ and } b_2 = [\frac{y}{k}(a_3 - a_4)\frac{q^{13}}{q^{12}}].$$

The policy functions can be written as

$$\Pi_1 \begin{bmatrix} \hat{y}_t \\ \hat{c}_t \\ \hat{n}_t \end{bmatrix} = \Pi_2 \begin{bmatrix} \hat{k}_t \\ \hat{z}_t \end{bmatrix}$$

or

$$\begin{bmatrix} \hat{y}_t \\ \hat{c}_t \\ \hat{n}_t \end{bmatrix} = \Pi \begin{bmatrix} \hat{k}_t \\ \hat{z}_t \end{bmatrix}$$

where

$$\Pi = \Pi_1^{-1}\Pi_2;\ \Pi_1 = \begin{bmatrix} 1 & -a_1 & 0 \\ 0 & q^{12} & 0 \\ 1 & -1 & -(1-n)^{-1} \end{bmatrix};\ \Pi_2 = \begin{bmatrix} a_2 & a_3 \\ -q^{11} & -q^{13} \\ 0 & 0 \end{bmatrix}.$$

See the gauss code for the remaining details.