ECON 5110 Solutions to Problem Set #3

1. Replicate the bottom two graphs in Figure 7.4 of Farmer's text. Comment on the differences and how well they match the U.S. data.

   Answer. Begin with 5 equations that describe equilibrium in the economy:

$$\frac{1}{c_t} = \beta E_t[\frac{1}{c_{t+1}}(1-\delta+m\frac{y_{t+1}}{k_{t+1}})] \tag{1}$$

$$k_{t+1} = y_t + (1-\delta)k_t - c_t \tag{2}$$

$$n\frac{y_t}{\ell_t} = c_t \ell_t^{\chi} \tag{3}$$

$$s_t = s_{t-1}^{\rho}\epsilon_t \tag{4}$$

$$y_t = s_t k_t^{\mu}\ell_t^{\nu} \tag{5}$$

where $\phi = 1$. The system of steady-state equations is

$$1 = \beta(1-\delta+m\frac{y}{k})$$

$$k = y + (1-\delta)k - c$$

$$n\frac{y}{\ell} = c\ell^{\chi}$$

$$y = k^{\mu}\ell^{\nu}.$$

Solving for ratios, we get

$$\frac{y}{k} = \frac{1-\beta(1-\delta)}{m\beta}$$

$$\frac{c}{k} = \frac{y}{k} - \delta$$

$$\ell = (n\frac{y}{c})^{\frac{1}{\chi+1}}.$$

Substiting y_t out of (1), (2) and (3) and ignoring technology shocks gives

$$\frac{1}{c_t} = \beta E_t[\frac{1}{c_{t+1}}(1-\delta+mk_{t+1}^{\mu-1}\ell_{t+1}^{\nu})] \tag{6}$$

$$k_{t+1} = k_t^{\mu}\ell_t^{\nu} + (1-\delta)k_t - c_t \tag{7}$$

$$\ell_t^{1+\chi-\nu} = \frac{nk_t^{\mu}}{c_t}. \tag{8}$$

The linearized versions of (6), (7), (8) are

$$
\begin{aligned}
\widehat{c}_t &= E_t\widehat{c}_{t+1} + a_1 E_t\widehat{k}_{t+1} + a_2 E_t\widehat{\ell}_{t+1} \\
\widehat{k}_{t+1} &= a_3\widehat{k}_t + a_4\widehat{\ell}_t + a_5\widehat{c}_t \\
\widehat{\ell}_t &= a_6\widehat{c}_t + a_7\widehat{k}_t.
\end{aligned}
$$

This can be reduced to a system of 2 equations

$$
\begin{aligned}
\widehat{c}_t &= E_t\widehat{c}_{t+1}(a_2a_6 + 1) + E_t\widehat{k}_{t+1}(a_2a_7 + a_1) \\
\widehat{k}_{t+1} &= \widehat{k}_t(a_3 + a_4a_7) + \widehat{c}_t(a_5 + a_4a_6)
\end{aligned}
$$

where the coefficients are defined as

$$
\begin{aligned}
a_1 &= -\frac{y}{k}\beta m(\mu - 1) \\
a_2 &= -\frac{y}{k}\beta m\nu \\
a_3 &= \frac{y\mu}{k} + (1 - \delta) \\
a_4 &= \nu\frac{y}{k} \\
a_5 &= -\frac{c}{k} \\
a_6 &= -\frac{y}{c}\frac{n}{(\chi + 1 - \nu)\ell^{\chi+1}} \\
a_7 &= -\mu a_6.
\end{aligned}
$$

Writing the system in standard matrix form where ϵ_{t+1} is a sunspot gives

$$
\begin{bmatrix} 0 & 1 \\ a_3 + a_4a_7 & a_5 + a_4a_6 \end{bmatrix} \begin{bmatrix} \widehat{k}_t \\ \widehat{c}_t \end{bmatrix} = \begin{bmatrix} a_2a_7 + a_1 & 1 + a_2a_6 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \widehat{k}_{t+1} \\ \widehat{c}_{t+1} \end{bmatrix} + \begin{bmatrix} 0 \\ \epsilon_{t+1} \end{bmatrix}
$$

or written more compactly as

$$
Y_t = AY_{t+1} + \epsilon_{t+1}
$$

where

$$A = \begin{bmatrix} 0 & 1 \\ a_3 + a_4 a_7 & a_5 + a_4 a_6 \end{bmatrix}^{-1} \begin{bmatrix} a_2 a_7 + a_1 & 1 + a_2 a_6 \\ 1 & 0 \end{bmatrix}.$$

Rewriting in matrix form gives

$$\begin{bmatrix} \widehat{k}_{t+1} \\ \widehat{c}_{t+1} \end{bmatrix} = A^{-1} \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix} + \begin{bmatrix} 0 \\ \epsilon_{t+1} \end{bmatrix}.$$

The evolution of the remaining variables are determined through the following functions

$$\pi_1 \begin{bmatrix} \widehat{x}_t \\ \widehat{y}_t \\ \widehat{\ell}_t \end{bmatrix} = \pi_2 \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix}.$$

Linearizing the production function (5) gives

$$\widehat{y}_t - \nu \widehat{\ell}_t = \mu \widehat{k}_t.$$

The linearized resource constraint is

$$\widehat{y}_t - \delta \frac{k}{y} \hat{x}_t = \frac{c}{y} \hat{c}_t.$$

The linearized labor-leisure tradeoff is

$$\widehat{\ell}_t = a_6 \widehat{c}_t + a_7 \widehat{k}_t.$$

This produces

$$\begin{bmatrix} 0 & 1 & -\nu \\ -\delta \frac{k}{y} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_t \\ \hat{y}_t \\ \hat{\ell}_t \end{bmatrix} = \begin{bmatrix} \mu & 0 \\ 0 & \frac{c}{y} \\ a_7 & a_6 \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix}.$$

2. Blanchard and Fischer, Exercise 5.6.

<u>Answer</u>: Equation (35) in Blanchard and Fischer is

$$y_t = k + aE(y_{t+1}|t) + \upsilon_t \tag{9}$$

where now it is assumed that y_t and υ_t are in the agent's information set at time t. First, solve the model for the fundamental solution by guessing the form of the solution as

$$y_t = \xi_0 + \xi_1 y_{t-1} + \upsilon_t + \xi_2 \nu_{t-1}. \tag{10}$$

Using (10) to generate expectations and then substituting into (9) gives

$$y_t = k + a[\xi_0 + \xi_1 y_t + \xi_2 \nu_t] + \upsilon_t,$$

which after rearranging gives

$$y_t = \left(\frac{k + a\xi_0}{1 - a\xi_1}\right) + \left(\frac{a\xi_2 + 1}{1 - a\xi_1}\right)\nu_t.$$

Therefore, we know that $\xi_1 = \xi_2 = 0$ and $\xi_0 = k/(1-a)$, which produces the fundamental solution

$$\bar{y}_t = \frac{k}{1-a} + \nu_t.$$

Under learning, assume agents form expectations using the PLM

$$y_t = \delta_n + \beta_n \nu_t.$$

Plugging the expectations from the PLM into (9), we get the ALM

$$y_t = (k + a\delta_n) + \upsilon_t.$$

From Evans and Honkapohja, we know that the fundamental solution $\bar{\theta} = (k/(1-a), 1)$ will be stable under learning if and only if

$$\frac{d\theta}{d\tau} = \theta^* - \theta = (k + (a-1)\delta_n, 1 - \beta_n)$$

is stable around $\bar{\theta}$, where $\theta^* = (k + a\delta_n, 1)$ and $\theta = (\delta_n, \beta_n)$. Therefore, the REE is stable under learning since we assume that $|a| < 1$.