Econ 5110 Solutions to the Midterm Exam

Spring 2009

Ramsey Model (100 pts). Consider a discrete-time version of the Ramsey model with a large number of identical agents who inelastically supply their labor (i.e., $l_t = 1$). The representative agent maximizes

$$E_0^* \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

by choosing $\{c_t\}_{t=0}^{\infty}$ subject to

$$k_{t+1} = y_t - c_t$$

where $y_t = s_t A_t k_t^{\alpha}$, $\alpha < 1$, and there is full depreciation. Technology shocks are random with zero persistence $s_t = \exp(\epsilon_t)$, $\epsilon_t \sim iid(0, \sigma^2)$. The A_t term represents an economy-wide production externality ($A_t = k_t^{\theta}$) that individuals take as given when making consumption decisions. Initial capital k_0 is given. Begin by assuming that $\theta = 0$.

1. (20 pts) Calculate the Euler equation for consumption, c_t. Provide some economic intuition for the equation.

   Answer. The Euler equation is

   $$\frac{1}{c_t} = \beta E_t^* \left(\frac{1}{c_{t+1}} \alpha k_{t+1}^{\alpha - 1} \right).$$

   Agents consume up to the point that the marginal utility of consumption today is equal to the discounted expected marginal utility of consumption from tomorrow's return on savings.

2. (20 pts) Find the steady-state equilibrium for (c, y, k) in terms of the fundamental parameters. Differentiate the steady-state expressions for (c, y, k) with respect to β and determine the signs of the derivatives. Do the signs make sense? Explain.

   Answer. The steady-state values for the variables are

   $$\begin{aligned} k &= (\alpha\beta)^{1/(1-\alpha)} \\ y &= k^{\alpha} = (\alpha\beta)^{\alpha/(1-\alpha)} \\ c &= y - k = (\alpha\beta)^{1/(1-\alpha)}[(\alpha\beta)^{\alpha} - 1]. \end{aligned}$$

   The derivatives are

   $$\begin{aligned} \frac{\partial k}{\partial \beta} &= \alpha^{1/(1-\alpha)}[1/(1-\alpha)]\beta^{(1/(1-\alpha))-1} = [1/((1-\alpha)\beta)]k > 0 \\ \frac{\partial y}{\partial \beta} &= \frac{\partial y}{\partial k}\frac{\partial k}{\partial \beta} = \alpha k^{\alpha-1}\frac{\partial k}{\partial \beta} > 0 \\ \frac{\partial c}{\partial \beta} &= \frac{\partial y}{\partial k}\frac{\partial k}{\partial \beta} - \frac{\partial k}{\partial \beta} = \frac{\partial k}{\partial \beta}(\frac{\partial y}{\partial k} - 1) = \frac{\partial k}{\partial \beta}(\alpha k^{\alpha-1} - 1) = \frac{\partial k}{\partial \beta}(\frac{1}{\beta} - 1) > 0. \end{aligned}$$

Yes, these signs make sense. More patience leads to more investment and higher steady-state capital, output and consumption.

3. (20 pts) Linearize the Euler equation (in terms of c_t, c_{t+1} and k_{t+1}), the production function (in terms of y_t, ϵ_t and k_t) and the law of motion for capital (in terms of k_{t+1}, y_t and c_t). Collapse the system to one second-order, stochastic linear difference equation in k and ϵ.

   Answer. The linearized equations are

$$\begin{aligned}\hat{c}_t &= E_t\hat{c}_{t+1} + (1-\alpha)\hat{k}_{t+1} \\ \hat{y}_t &= \alpha\hat{k}_t + \epsilon_t \\ \hat{y}_t &= (k/y)\hat{k}_{t+1} + (c/y)\hat{c}_t.\end{aligned}$$

   Collapsing to one equation in k and ϵ gives

$$\hat{k}_t = a\hat{k}_{t+1} + bE_t\hat{k}_{t+2} + c\epsilon_t,$$

   where

$$a = \frac{[(k/y) + \alpha + (c/y)(1-\alpha)]}{\alpha}, b = \frac{-(k/y)}{\alpha} \text{ and } c = -\frac{1}{\alpha}.$$

4. (20 pts) Find the equilibrium under naive expectations and describe the transition dynamics to a single positive technology shock.

   Answer. The Euler equation for consumption becomes

$$\frac{1}{c_t} = \beta\left(\frac{1}{c_t}\alpha k_t^{\alpha-1}\right).$$

   This implies that

$$\begin{aligned}k_t &= k \\ y_t &= s_t k^\alpha \\ c_t &= y_t - k\end{aligned}$$

   for all t. Written in proportional deviation from steady state, we have

$$\begin{aligned}\hat{k}_t &= 0 \\ \hat{y}_t &= \epsilon_t \\ \hat{c}_t &= (y/c)\epsilon_t.\end{aligned}$$

With naive expectations, the effect of the technology shock lasts one period. The capital stock remains at steady state. Output and consumption go up by the same amount.

5. (20 pts) Find the rational expectations equilibrium (REE) in the presence of a production externality ($\alpha + \theta > 1$) and discuss the nature of the equilibrium.

Answer. In this case there are increasing returns to scale at the social level and we have an irregular model (Farmer, 1999). This type of model displays multiple equilibria, indeterminancy and can support sunspot equilibrium. The REE is given by

$$\begin{aligned} \hat{c}_t &= \hat{c}_{t+1} + (1 - \alpha - \theta)\hat{k}_{t+1} + w^c_{t+1} \\ \hat{y}_t &= \alpha\hat{k}_t + \epsilon_t \\ \hat{y}_t &= (k/y)\hat{k}_{t+1} + (c/y)\hat{c}_t, \end{aligned}$$

where w^c_{t+1} is a sunspot such that $E_t w^c_{t+1} = 0$.

Bonus. (10 pts). The rational expectations equilibrium (REE) is of the form: $\hat{k}_{t+1} = \phi_1\hat{k}_t + \phi_2\epsilon_t$. Use undetermined coefficients to solve for ϕ_1 and ϕ_2 in terms of the fundamental parameters.

Answer. Using undetermined coefficients, we have

$$\begin{aligned} \phi_1 &= \frac{1}{a + b\phi_1} \\ \phi_2 &= \frac{-c}{a + b\phi_1}. \end{aligned}$$

The solutions are

$$\begin{aligned} \phi_1 &= \frac{-a \pm \sqrt{a^2 + 4b}}{2b} \\ \phi_2 &= \frac{-c}{a + b\phi_1}. \end{aligned}$$