ECON 5110 Class Notes
Real Business Cycle Theory: Solow and Ramsey Growth Models

1 Introduction

The Solow and Ramsey growth models are the backbone of modern business cycle theory. In fact, real business cycle (RBC) theory, as initiated by Long and Plosser (1983) and Kydland and Prescott (1982), is simply a Ramsey neoclassical growth model with stochastic technology shocks. Modern New Keynesian models also share many of the features of the Solow and Ramsey growth models.

2 Solow Growth Model

2.1 The Basics

The Solow growth model begins with a constant returns to scale (CRS) production function

$$Y(t) = F(K(t), A(t)L(t)) \tag{1}$$

where the terms are defined as

- $Y(t)$ – output
- $K(t)$ – capital stock
- $A(t)$ – knowledge
- $L(t)$ – labor.

When A and L enter (1) multiplicatively, technological progress is said to be labor-augmenting, which implies that the K/Y ratio will be constant in the steady state. It will be convenient to use the CRS property to rewrite the production function in its intensive form by multiplying all terms by $1/AL$

$$\frac{Y}{AL} = F(\frac{K}{AL}, 1) \Rightarrow y = f(k) \tag{2}$$

where $y = Y/AL$ and $k = K/AL$ are output and capital per unit of effective labor. In addition to being CRS, the production function is assumed to satisfy $f(0) = 0$, $f'(k) > 0$, $f''(k) < 0$ and the Inada conditions

($\lim_{k\to 0} f'(k) = \infty$ and $\lim_{k\to\infty} f'(k) = 0$). It will be convenient to work with the Cobb-Douglas production function

$$Y(t) = K(t)^{\alpha}(A(t)L(t))^{1-\alpha} \tag{3}$$

which in its intensive form is

$$y(t) = k(t)^{\alpha}.$$

2.2 Dynamics

The model is set in continuous time. Labor (L) and knowledge (A) are assumed to be exogenous and grow at exponential rates according to

$$\begin{aligned} \dot{L}(t) &= nL(t) \Rightarrow L(t) = e^{nt}L(0) \\ \dot{A}(t) &= gA(t) \Rightarrow A(t) = e^{gt}A(0). \end{aligned}$$

By assumption, a fraction s of output is devoted toward savings (and hence investment) and a fraction $(1-s)$ of output is devoted toward consumption. Therefore, s is the exogenous saving rate. Capital evolves according to

$$\dot{K}(t) = sY(t) - \delta K(t) \tag{4}$$

where $0 \leq \delta \leq 1$ is the depreciation rate. Writing (4) in its intensive form gives

$$\dot{k}(t) = sk(t)^{\alpha} - (n + g + \delta)k(t). \tag{5}$$

The evolutions for $y(t) = k(t)^{\alpha}$ and $c(t) = (1-s)y(t)$ can be found easily from (5).

2.3 Steady State

In the steady state, capital per unit of effective labor is constant (i.e., $\dot{k}(t) = 0$). From (5), this implies that total savings (per unit of effective labor) $sk(t)^{\alpha}$ must equal break-even investment $(n + g + \delta)k(t)$. It is common to denote this steady-state level of capital as k^*. If $k(t) > k^*$, then because capital exhibits diminishing marginal returns, savings will not be sufficient to replace the capital lost to depreciation, labor growth and technological progress. As a result, the capital stock will fall ($\dot{k}(t) < 0$) and return to k^*. The opposite is true if $k(t) < k^*$. This implies that the system is stable. See Figure 1.2 on page 15 in Romer.

2.4 Balanced Growth Path

For any initial value of $k(0)$, the economy will eventually settle down to the steady-state level k^*. Since $k^* = K^*/AL$, this implies that in the steady state, the actual capital stock (not per unit of effective labor) will be growing at rate $n+g$. Furthermore, since AL also grows at rate $n+g$, this implies that Y will grow at rate $n+g$. We often use output per person Y/L as a measure of standards of living across time and countries. Y/L will be growing at rate g along the balanced growth path.

2.5 Conclusions and Shortcomings

I begin by highlighting the major findings of the Solow model:

1. For any initial $k(0)$, the economy will converge to a balanced growth path where Y/L will grow at the exogenous rate of technological progress, g.

2. There is a unique saving rate, s_g, which maximizes consumption per worker, C/L. This often referred to as the golden-rule saving rate and is given by the condition $f'(k^*) = n + g + \delta$.

3. The model predicts conditional convergence. That is, once you control for differences in savings rates, depreciation and labor growth rates, poorer countries with smaller capital stocks should grow faster and eventually catch up with richer countries. The empirical evidence supporting conditional converge across countries is mixed at best.

4. Growth accounting can be used to decompose output growth into the parts coming from capital accumulation and technological progress. Total differentiation of (3) along with some minor algebra gives
$$\frac{\dot{Y}(t)}{Y(t)} = \alpha \frac{\dot{K}(t)}{K(t)} + (1-\alpha)\frac{\dot{L}(t)}{L(t)} + R(t)$$
where $R(t)$ is often referred to as the Solow residual or total factor productivity.

There are several well-documented shortcomings of the Solow model:

1. Steady-state growth in Y/L is entirely exogenous. In response, a new endogenous growth paradigm (led by Paul Romer) has emerged.

2. The micro fundamentals of household and firm decisions are assumed away. The Ramsey model below addresses this shortcoming.

3. The Solow model, for reasonable parameterizations, is unable to explain the vast differences in living standards across countries or across time. It requires unrealistically large differences (either across time or countries) in capital-labor ratios and/or technology.

3 Ramsey Growth Model

The Ramsey model extends the Solow model to allow for explicitly optimal behavior by firms and households.

3.1 The Basics

Begin by assuming the following:

- There are a large number of identical households with each member supplying one unit of labor.
- For simplicity, the initial amount of labor is set at unity ($L(0) = 1$) and there is no labor growth ($n = 0$).
- Households own the firms.
- Each firm hires labor and rent capital in competitive input markets. Each firm sells its output in a competitive output market.
- Each firm has access to the CRS production function (3) with $A(t)$ growing at rate g and $A(0) = 1$.
- There is no depreciation of capital (i.e., $\delta = 0$).

3.2 Household Behavior

Households are assumed to be infinitely lived and maximize a discounted stream of future utility given by

$$\int_{t=0}^{\infty} e^{-\rho t} u[C(t)]dt$$

by choosing $C(t)$, the control variable, at each point in time. The instantaneous utility function $u[C(t)]$ is assumed to be in the constant elasticity of substitution (CES) class

$$u[C(t)] = \frac{C(t)^{1-\theta} - 1}{1-\theta}$$

where $\theta = -Cu''/u' > 0$ is called the coefficient of relative risk aversion. Larger θs indicate more curvature in the utility function and less willingness to substitute consumption intertemporally. $\sigma = 1/\theta$ is commonly referred to as the intertemporal elasticity of substitution. As $\theta \to 0$, the utility function becomes linear in consumption and $\sigma \to \infty$.

The households' constraints are given by (i) the flow budget constraint

$$\dot{a}(t) = w(t) + r(t)a(t) - C(t),$$

where $a(t)$ is the sole asset, $w(t)$ is the wage rate and $r(t)$ is the asset rate of return; (ii) the no Ponzi-game condition

$$\lim_{t\to\infty} e^{-R(t)}a(t) \geq 0$$

$R(t) = \int_{s=0}^{t} r(s)ds$; and (iii) $a(0)$ given. Solving this continuous-time dynamic problem involves using calculus of variations (no derivations will be provided at this time). Begin by writing down the present-value Hamiltonian

$$H = e^{-\rho t}[\frac{C(t)^{1-\theta} - 1}{1-\theta}] + \lambda(t)[w + r(t)a(t) - C(t)]$$

where $\lambda(t)$, the costate variable, is the present-value shadow price of income. The first-order conditions for maximization are

$$\frac{\partial H}{\partial C} = 0 \Rightarrow \lambda(t) = e^{-\rho t}C(t)^{-\theta} \quad (6)$$

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial a} \Rightarrow \dot{\lambda}(t) = -\lambda(t)r(t) \quad (7)$$

and the transversality condition is

$$\lim_{t\to\infty} \lambda(t)a(t) = 0.$$

Combining (6) and (7), we get

$$\frac{\dot{C}(t)}{C(t)} = \frac{r(t) - \rho}{\theta}. \quad (8)$$

Equation (8) is known as the Euler equation for consumption.

3.3 Firm Behavior

Firm behavior is simpler. Firms choose labor and capital to maximize profits per period

$$\Pi(t) = F(K(t), A(t)L(t)) - r(t)K(t) - w(t)L(t),$$

which written in its intensive form is

$$\Pi(t) = e^{gt}[f(k(t)) - r(t)k(t) - w(t)e^{-gt}].$$

Since firms take $r(t)$ and $w(t)$ as given, they will rent capital up to the point that its marginal product equals rental rate or

$$r(t) = f'(k(t)). \tag{9}$$

This will result in zero economic profits if labor is also paid its marginal product or

$$w(t) = e^{gt}[f(k(t)) - k(t)f'(k(t))].$$

3.4 Equilibrium Dynamics and Welfare

Equilibrium dynamics for this economy are given by the capital accumulation equation, (8) and (9). Letting $C(t) = e^{gt}c(t)$ and $a(t) = e^{gt}k(t)$, along with the appropriate substitutions, the equilibrium for this economy reduces to

$$\begin{aligned} \frac{\dot{c}(t)}{c(t)} &= \frac{f'(k(t)) - \rho - \theta g}{\theta}, \\ \dot{k}(t) &= f(k(t)) - c(t) - gk(t), \end{aligned}$$

along with the transversality condition and $k(0)$ given. The dynamic properties of this economy have been studied extensively. Figure 2.4 on page 58 of Romer depicts the dynamics in a phase diagram. The primary conclusions are

1. There exists a unique (c, k) combination that produces steady-state ($\dot{k} = \dot{c} = 0$) growth. This is given by point E in Figure 2.4 of Romer.

2. For a given $k(0)$, there is a unique $c(0)$ that will result in a non-divergent path to the steady state. This path is known as the saddle path and this general property is known as saddle-path stability.

3. Since markets are competitive and there are no externalities, the first welfare theorem of economics states that this competitive equilibrium is Pareto optimal (i.e., no agent can be made better without making another worse off). This is also the same outcome that would be reached by a benevolent social planner that treated all agents equally.

4. The steady-state level of consumption per worker in the Ramsey model (sometimes referred to as the modified golden-rule level) is less than the golden-rule level derived from the Solow model. This happens because impatient optimizing agents are willing to trade off a permanently lower level of future consumption fo a higher level of consumption today

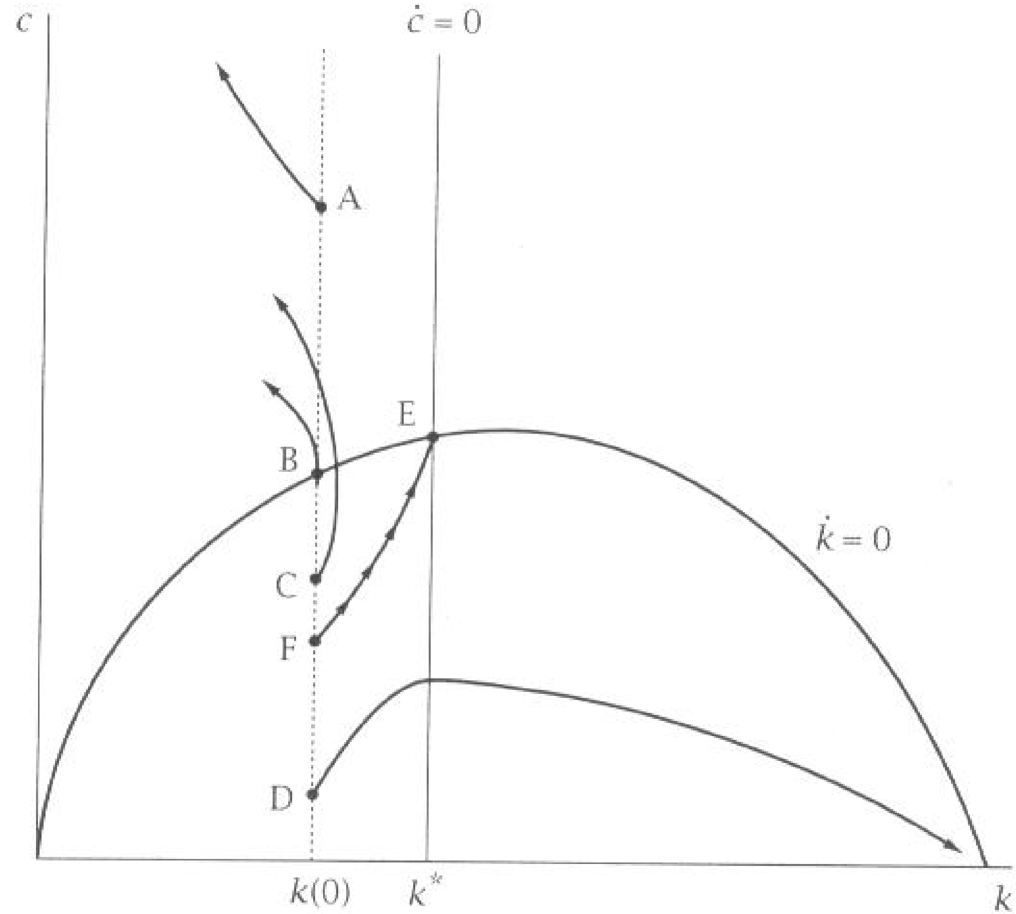

FIGURE 2.4 The behavior of c and k for various initial values of c