ECON 5350 Solutions to Problem Set #1

1. Suppose that a lottery ticket costs \$1 per play. The game is played by drawing six numbers (order matters) without replacement from the numbers 1 to 48. If you guess all six numbers, you win the prize (assume only one player wins). Now suppose that N equals the number of tickets sold and P equals the size of the prize. N and P are related by

$$\begin{aligned} N &= 5 + 1.2P \\ P &= 1 + 0.4N \end{aligned}$$

N and P are measured in millions. What is the expected value of a ticket in this game?

Answer. The expected value of the lottery ticket is given by

$$\Pr(\text{Win})(P - \$1) + \Pr(\text{Lose})(-\$1).$$

Solving the linear system in P and N gives $P = \$5,769,000$, and the probability of drawing a winning ticket is

$$\frac{1}{48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 \cdot 43} = 1.132 \times 10^{-10}.$$

The expected value is thus

$$(1.132 \times 10^{-10})(\$5,768,999) + (1 - 1.132 \times 10^{-10})(-\$1) = -\$0.9993.$$

2. Chebyshev's Inequality. For the following two probability distributions, find the lower limit of the probability of the indicated event using Chebyshev's inequality and the exact probability using the appropriate table.

Answers.

(a) $X \sim N(0, 3^2)$ and $-4 < X < 4$. Using Table G.1 from Greene

$$\Pr(-4/3 < Z < 4/3) = 1 - 2(1 - 0.9082) = 0.8164.$$

Using Chebyshev's inequality with $k = 4/3$, we get

$$\Pr(-4 < X < 4) \geq 0.4375.$$

(b) $X \sim \chi^2(8)$ and $0 < X < 16$. Using Table G.3 from Greene, the $\Pr(X < 16)$ is between 0.950 and

0.975. Using Chebyshev's inequality with $k = 2$,

$$\Pr(X < 16) \geq 0.75.$$

3. Use the following joint probability distribution

		X		
		0	1	2
	0	0.05	0.1	0.03
Y	1	0.21	0.11	0.19
	2	0.08	0.15	0.08

to complete the following

<u>Answers</u>.

(a) Compute $\Pr(Y < 2)$, $\Pr(Y < 2, X > 0)$ and $\Pr(Y = 1, X \geq 1)$.

- $\Pr(Y < 2) = 0.18 + 0.51 = 0.69$.
- $\Pr(Y < 2, X > 0) = 0.1 + 0.03 + 0.11 + 0.19 = 0.43$.
- $\Pr(Y = 1, X \geq 1) = 0.11 + 0.19 = 0.30$.

(b) Find the marginal distributions of X and Y.

- $f_x(x = 0) = 0.34$, $f_x(x = 1) = 0.36$ and $f_x(x = 2) = 0.30$.
- $f_y(y = 0) = 0.18$, $f_y(y = 1) = 0.51$ and $f_y(y = 2) = 0.31$.

(c) Calculate $E(X)$, $E(Y)$, $Var(X)$, $Var(Y)$, $Cov(X, Y)$ and $E(X^2Y^3)$.

- $E(X) = 1(0.36) + 2(0.30) = 0.96$.
- $E(Y) = 1(0.51) + 2(0.31) = 1.13$.
- $Var(X) = (-0.96)^2 0.34 + (1 - 0.96)^2 0.36 + (2 - 0.96)^2 0.30 = 0.6384$.
- $Var(Y) = (-1.13)^2 0.18 + (1 - 1.13)^2 0.51 + (2 - 1.13)^2 0.31 = 0.47$.
- $E(XY) = 1(0.11) + 2(0.19) + 2(0.15) + 4(0.08) = 1.11$.
- $Cov(X, Y) = E(XY) - \mu_x \mu_y = 1.11 - 0.96(1.13) = 0.0252$.
- $E(X^2Y^3) = 1(0.11) + 4(0.19) + 8(0.15) + 32(0.08) = 4.63$.

(d) Calculate $Cov(Y, X^2)$.

- $E(X^2) = 1(0.36) + 4(0.30) = 1.56$.
- $E(X^2Y) = 1(0.11) + 4(0.19) + 2(0.15) + 8(0.08) = 1.81$.

- $Cov(X^2, Y) = E(X^2Y) - \mu_{x^2}\mu_y = 1.81 - (1.56)1.13 = 0.0472.$

(e) What are the conditional distributions of Y given $X = 2$ and of X given $Y > 0$?

- $f_y(Y = 0|X = 2) = 0.03/0.30 = 0.10.$
- $f_y(Y = 1|X = 2) = 0.19/0.30 = 0.63.$
- $f_y(Y = 2|X = 2) = 0.08/0.30 = 0.27.$
- $f_x(X = 0|Y > 0) = 0.29/0.82 = 0.35.$
- $f_x(X = 1|Y > 0) = 0.26/0.82 = 0.32.$
- $f_x(X = 2|Y > 0) = 0.27/0.82 = 0.33.$

(f) Find $E(Y|X)$ and $Var(Y|X)$.

- $E(Y|X = 0) = 1(0.21/0.34) + 2(0.08/0.34) = 1.09.$
- $E(Y|X = 1) = 1(0.11/0.36) + 2(0.15/0.36) = 1.14.$
- $E(Y|X = 2) = 1(0.19/0.30) + 2(0.08/0.30) = 1.17.$
- $E(Y^2|X = 0) = 1(0.21/0.34) + 4(0.08/0.34) = 1.56.$
- $E(Y^2|X = 1) = 1(0.11/0.36) + 4(0.15/0.36) = 1.97.$
- $E(Y^2|X = 2) = 1(0.19/0.30) + 4(0.08/0.30) = 1.70.$
- $Var(Y|X = 0) = 1.56 - (1.09)^2 = 0.37.$
- $Var(Y|X = 1) = 1.97 - (1.14)^2 = 0.67.$
- $Var(Y|X = 2) = 1.70 - (1.17)^2 = 0.33.$

4. Find the MGF of the exponential and Poisson distributions.

Answer.

(a) The exponential MGF is

$$M(t) = \int_0^\infty e^{tx}\theta e^{-\theta x}dx = \int_0^\infty e^{(t-\theta)x}\theta dx = \theta\left(\frac{1}{t-\theta}e^{(t-\theta)x}|_{x=0}^{\infty}\right) = \frac{\theta}{\theta - t} \text{ for } t < \theta.$$

(b) The Poisson MGF is

$$M(t) = \sum_{x=0}^{\infty}\left(\frac{e^{tx}e^{-\lambda}\lambda^x}{x!}\right) = \sum_{x=0}^{\infty}\left(\frac{(e^t\lambda)^x e^{-\lambda}}{x!}\right) = e^{-\lambda + e^t\lambda}\sum_{x=0}^{\infty}\left(\frac{(e^t\lambda)^x e^{-(e^t\lambda)}}{x!}\right).$$

The last summation is the sum of probabilities from a Poisson pdf with parameter $e^t\lambda$. Therefore, $M(t) = e^{\lambda(e^t-1)}$.

5. Use the MGFs to find the mean and variance of the exponential and Poisson distributions.

Answer.

(a) Exponential mean and variance.

$$\begin{aligned}
\mu &= \frac{dM(t)}{dt}|_{t=0} = (\theta - t)^{-2}\theta|_{t=0} = \frac{1}{\theta}. \\
\sigma^2 &= \frac{d^2M(t)}{dt^2}|_{t=0} - \frac{1}{\theta^2} = 2(\theta - t)^{-3}\theta|_{t=0} - \frac{1}{\theta^2} = \frac{2}{\theta^2} - \frac{1}{\theta^2} = \frac{1}{\theta^2}.
\end{aligned}$$

(b) Poisson mean and variance.

$$\begin{aligned}
\mu &= \frac{dM(t)}{dt}|_{t=0} = \lambda e^t \exp\left(-\lambda(1 - e^t)\right)|_{t=0} = \lambda. \\
\sigma^2 &= \frac{d^2M(t)}{dt^2}|_{t=0} - \lambda^2 = \left\{[(\lambda e^t)^2 \exp(\lambda(-1 + e^t))] + [\lambda e^t \exp(\lambda(-1 + e^t))]\right\}|_{t=0} - \lambda^2 \\
&= \lambda^2 + \lambda - \lambda^2 = \lambda.
\end{aligned}$$

GAUSS PROBLEMS

6. Generate a $(n \times 1)$ vector of independent draws from a $N(0,1)$ distribution where $n = 100$. Calculate the sample mean $(\bar{X})$, sample variance(s^2) and a histogram of the sample pdf.

   Answer. See attached Gauss code.

7. Repeat Exercise 6 using a $\chi^2(1)$ distribution.

   Answer. See attached Gauss code.

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@ *********************** @
@ ECON 5350 PROBLEM SET #1 @
@ *********************** @

@ ********** @
@ PROBLEM #6 @
@ ********** @

nobs = 100;
x = rndn(nobs,1);
xbar = sumc(x)/nobs;
xdm = x - xbar;
xvar = sumc(xdm^2)/(nobs-1);
print "N(0,1) sample mean (n = " nobs ") = " xbar;
      "N(0,1) sample variance (n = " nobs ") = " xvar;

library pgraph;
graphset;
xlabel("X");
title("N(0,1) Sample Probability Density Function (n = 100)");
{b, m, f} = hist(x, 20);

@ ********** @
@ PROBLEM #7 @
@ ********** @

nobs = 100;
x = rndn(nobs,1);
chi = x.*x;
chibar = sumc(chi)/nobs;
chidm = chi - chibar;
chivar = sumc(chidm^2)/(nobs-1);
print "Chi square sample mean (n = " nobs ") = " chibar;
      "Chi square sample variance (n = " nobs ") = " chivar;

library pgraph;
graphset;
xlabel("X");
title("ChiSquare(1) Sample Probability Density Function (n = 100)");
{b, m, f} = hist(chi, 20);
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