ECON 5350 Solutions to Problem Set #3

1. Greene 5th edition, Exercise 14.4.

   Answer: The GLS estimator is

$$\hat{\beta} = \begin{bmatrix} \sigma^{11}X'X & \sigma^{12}X'X \\ \sigma^{12}X'X & \sigma^{22}X'X \end{bmatrix}^{-1} \begin{bmatrix} \sigma^{11}X'y_1 + \sigma^{12}X'y_2 \\ \sigma^{12}X'y_1 + \sigma^{22}X'y_2 \end{bmatrix}.$$

   The matrix to be inverted equals $[\Sigma^{-1} \otimes X'X]^{-1} = [\Sigma \otimes (X'X)^{-1}]$, where σ^{ij} and σ_{ij} refer to the elements of Σ^{-1} and Σ, respectively. Making the substitution produces

$$\begin{aligned} \hat{\beta} &= \begin{bmatrix} \sigma_{11}(X'X)^{-1} & \sigma_{12}(X'X)^{-1} \\ \sigma_{12}(X'X)^{-1} & \sigma_{22}(X'X)^{-1} \end{bmatrix} \begin{bmatrix} \sigma^{11}X'y_1 + \sigma^{12}X'y_2 \\ \sigma^{12}X'y_1 + \sigma^{22}X'y_2 \end{bmatrix} \\ &= \begin{bmatrix} \sigma_{11}(\sigma^{11}b_1 + \sigma^{12}b_2) + \sigma_{12}(\sigma^{12}b_1 + \sigma^{22}b_2) \\ \sigma_{12}(\sigma^{11}b_1 + \sigma^{12}b_2) + \sigma_{22}(\sigma^{12}b_1 + \sigma^{22}b_2) \end{bmatrix} \\ &= \begin{bmatrix} (\sigma_{11}\sigma^{11} + \sigma_{12}\sigma^{12})b_1 + (\sigma_{11}\sigma^{12} + \sigma_{12}\sigma^{22})b_2 \\ (\sigma_{12}\sigma^{11} + \sigma_{22}\sigma^{12})b_1 + (\sigma_{12}\sigma^{12} + \sigma_{22}\sigma^{22})b_2 \end{bmatrix} \\ &= \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \end{aligned}$$

   because the terms in the second to last line are the corresponding elements of $\Sigma\Sigma^{-1} = I$.

2. Greene 5th edition, Exercise 14.6.

   Answer: The GLS estimator is

$$\hat{\gamma} = (\hat{\alpha}_1, \hat{\beta}, \hat{\alpha}_2)' = \begin{bmatrix} \sigma^{11}i'i & \sigma^{11}i'x & \sigma^{12}i'i \\ \sigma^{11}x'i & \sigma^{11}x'x & \sigma^{12}x'i \\ \sigma^{21}i'i & \sigma^{21}i'x & \sigma^{22}i'i \end{bmatrix}^{-1} \begin{bmatrix} \sigma^{11}i'y_1 + \sigma^{12}i'y_2 \\ \sigma^{11}x'y_1 + \sigma^{12}x'y_2 \\ \sigma^{21}i'y_1 + \sigma^{22}i'y_2 \end{bmatrix} \quad (1)$$

   where σ^{ij} is the (i, j) element of the inverse of the covariance matrix of the errors and i is a column vector of ones. Using $\bar{x} = 0$ and simplifying, equation (1) can be written as

$$\begin{aligned}
\hat{\gamma} &= \begin{bmatrix} \sigma^{11}n & 0 & \sigma^{12}n \\ 0 & \sigma^{11}x'x & 0 \\ \sigma^{12}n & 0 & \sigma^{22}n \end{bmatrix}^{-1} \begin{bmatrix} \sigma^{11}n\bar{y}_1 + \sigma^{12}n\bar{y}_2 \\ \sigma^{11}x'y_1 + \sigma^{12}x'y_2 \\ \sigma^{21}n\bar{y}_1 + \sigma^{22}n\bar{y}_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sigma^{22}}{n(\sigma^{11}\sigma^{22}-(\sigma^{12})^2)} & 0 & \frac{-\sigma^{12}}{n(\sigma^{11}\sigma^{22}-(\sigma^{12})^2)} \\ 0 & \frac{1}{\sigma^{11}x'x} & 0 \\ \frac{-\sigma^{12}}{n(\sigma^{11}\sigma^{22}-(\sigma^{12})^2)} & 0 & \frac{\sigma^{11}}{n(\sigma^{11}\sigma^{22}-(\sigma^{12})^2)} \end{bmatrix} \begin{bmatrix} \sigma^{11}n\bar{y}_1 + \sigma^{12}n\bar{y}_2 \\ \sigma^{11}x'y_1 + \sigma^{12}x'y_2 \\ \sigma^{21}n\bar{y}_1 + \sigma^{22}n\bar{y}_2 \end{bmatrix} = \begin{bmatrix} \bar{y}_1 \\ \frac{x'y_1}{x'x} + \theta\frac{x'y_2}{x'x} \\ \bar{y}_2 \end{bmatrix}
\end{aligned}$$

where $\theta = \frac{\sigma^{12}}{\sigma^{11}}$. This shows that the GLS estimates of the intercepts are the same as the OLS estimates, and the GLS estimate of the slope is a weighted average of the two OLS slope estimates (with $\bar{x} = 0$).