1. **Generalized Least Squares (25 pts)**. Consider the following partitioned regression model:

\[
\begin{bmatrix}
Y_1 \\
Y_2
\end{bmatrix}
= \begin{bmatrix}
X_1 \\
X_2
\end{bmatrix} \beta + 
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2
\end{bmatrix} \tag{1}
\]

where

\[
\Omega = \begin{bmatrix}
\sigma_1^2 I_{n_1} & 0 \\
0 & \sigma_2^2 I_{n_2}
\end{bmatrix}
\]

is the variance covariance matrix of the error terms. Find the formula for the GLS estimator and suggest a feasible GLS estimator.

**Solution.** The GLS estimator is

\[
\hat{\beta}_{GLS} = \left( \frac{X'_1 X_1}{\sigma_1^2} + \frac{X'_2 X_2}{\sigma_2^2} \right)^{-1} \left( \frac{X'_1 Y}{\sigma_1^2} + \frac{X'_2 Y}{\sigma_2^2} \right)^{-1}.
\]

A feasible GLS estimator could be obtained by substituting

\[
\hat{\sigma}_j^2 = \frac{(Y_j - X_j b)'(Y_j - X_j b)}{n_j - k}
\]

into the GLS estimator, where \(b\) is the OLS estimator.

2. **Autocorrelation (30 pts)**. Consider the simple linear regression model \(y_t = \beta_0 + \beta_1 x_t + \epsilon_t\), where the errors exhibit second-order autocorrelation: \(\epsilon_t = \rho_2 \epsilon_{t-2} + \mu_t\).

(a) Calculate the autocovariance function (i.e., \(\gamma(s) = cov(\epsilon_t, \epsilon_{t-s})\)) under the assumption that \(\epsilon_t\) is a weakly covariance stationary process.

**Solution.** The variance is

\[
\gamma(0) = \sigma_\mu^2 / (1 - \rho_2^2).
\]

The autocovariance function is

\[
\gamma(s) = \rho_2^{s-1} \gamma(0)
\]

for \(s = \pm 2, \pm 4, \pm 6, \ldots\) and zero otherwise.

(b) Assuming \(\rho_2\) is known, derive the GLS estimator of \(\beta_1\).

**Solution.** The GLS estimator involves estimating the following equation via OLS:

\[
(y_t - \rho_2 y_{t-2}) = \beta_0 (1 - \rho_2) + \beta_1 (x_t - \rho_2 x_{t-2}) + \mu_t
\]
for $t = 3, 4, ..., T$. The first two observations can either be dropped or scaled in such a way that the variance of the transformed dependent variable equals $\sigma^2_{u_t}$. This can be calculated given the information in part (a).

(c) Describe how one would calculate an asymptotically efficient two-step estimator of $\beta_1$ if $\rho_2$ were unknown.

**Solution.** First we would find a consistent estimate of $\rho_2$ using an estimator such as

$$\hat{\rho}_2 = \frac{\sum_{t=3}^{T} e_t e_{t-2}}{\sum_{t=1}^{T} e_t^2}.$$ 

Then we would substitute $\hat{\rho}_2$ into the equation above and estimate via OLS.

3. **SUR and Panel-Data Models (45 pts).** Consider the following SUR system:

$$
\begin{bmatrix}
Y_1 \\
Y_2
\end{bmatrix} =
\begin{bmatrix}
X_1 & 0 \\
0 & X_2
\end{bmatrix} 
\begin{bmatrix}
\beta_1 \\
\beta_2
\end{bmatrix} + 
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2
\end{bmatrix},
$$

where $Y_1$, $Y_2$, $\epsilon_1$ and $\epsilon_2$ are of dimension $(T \times 1)$; $X_j$ is of dimension $(T \times k_j)$ for $j = 1, 2$; $\beta_j$ is of dimension $(k_j \times 1)$ for $j = 1, 2$; and the system variance-covariance matrix is

$$\Omega = \Sigma \otimes I_T = 
\begin{bmatrix}
\sigma^2_{1T} & \sigma_{12T} \\
\sigma_{12T} & \sigma^2_{2T}
\end{bmatrix}$$

where

$$\Omega^{-1} = \Sigma^{-1} \otimes I_T = 
\begin{bmatrix}
a_{11T} & a_{12T} \\
a_{12T} & a_{22T}
\end{bmatrix}.$$

(a) The SUR system above could be used to incorporate panel data. Describe an estimation strategy to obtain consistent, asymptotically efficient estimates for a fixed effects version of Model (2).

**Solution.** We could think of each equation as a cross section of length $T$. If there were $n$ cross sections, then we would need $n$ equations. The first column of each $X_j$ matrix would need to be filled with ones to capture the fixed effects. The first step in obtaining a consistent and asymptotically efficient estimate is to use the OLS residuals to find the following consistent estimate for $\Sigma$:

$$\hat{\Sigma} = \frac{1}{T} 
\begin{bmatrix}
e_1'e_1 & e_1'e_2 \\
e_2'e_1 & e_2'e_2
\end{bmatrix}.$$
This estimate can then be substituted into the GLS formula

$$\hat{\beta}_{GLS} = [X'\Omega^{-1}X]^{-1}[X'\Omega^{-1}Y].$$

The first element for each $\hat{\beta}_j$ would be the corresponding fixed effect.

(b) Show that equation-by-equation OLS is equivalent to GLS when $X_1 = X_2$. For simplicity, consider the case when $k_1 = k_2 = 1$ and $T = 2$.

Solution. The GLS formula is

$$\hat{\beta}_{GLS} = [X'\Omega^{-1}X]^{-1}[X'\Omega^{-1}Y] = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ 0 & a_{11} & 0 & a_{12} \\ a_{12} & 0 & a_{22} & 0 \\ 0 & a_{12} & 0 & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix}^{-1} \times \begin{bmatrix} x_1 \\ x_2 \\ y_{11} \\ y_{12} \\ y_{21} \\ y_{22} \end{bmatrix}$$

$$= \begin{bmatrix} x_1^2a_{11} + x_2^2a_{12} & x_1^2a_{11} + x_2^2a_{12} \\ x_1^2a_{12} + x_2^2a_{22} & x_1^2a_{12} + x_2^2a_{22} \end{bmatrix}^{-1} \times \begin{bmatrix} x_1y_{11} + x_2y_{12} + x_1y_{11}a_{11} + x_2y_{12}a_{11} + x_1y_{21}a_{12} + x_2y_{22}a_{12} \\ x_1y_{11} + x_2y_{12} + x_1y_{21}a_{12} + x_2y_{22}a_{12} \end{bmatrix}$$

$$= \begin{bmatrix} 1/(x_1^2 + x_2^2)\Sigma^{-1} \end{bmatrix} \begin{bmatrix} x_1y_{11} + x_2y_{12} \\ x_1y_{21} + x_2y_{22} \end{bmatrix} = b_{OLS}.$$ 

(c) Describe, in detail, how to test Model (1) versus Model (2) when $\sigma_{12} = 0$. Continue to assume that $X_1 = X_2$, $k_1 = k_2 = 1$ and $T = 2$.

Solution. This can be done as a straightforward $F$ test. If we consider Model (1) as the restricted system and Model (2) as the unrestricted system, then we can estimate both with OLS. Using the restricted ($e_r$) and unrestricted ($e$) residuals, we form

$$F = \frac{(e'_re_r - e'e)/1}{(e'e)/(4-2)}.$$
which has an $F$ distribution with one numerator degrees of freedom and two denominator degrees of freedom. The null and alternative hypotheses are

$$
H_0 \ : \ \beta_1 = \beta_2 \\
H_A \ : \ \beta_1 \neq \beta_2.
$$