Home work # 4
Math 5290-Fall 2008
Due date 10/22/08 in class

1. **(ex19.2 pp171).** Let $(X_n)_{n\geq 1}$ be $\mathcal{N}(\mu_n, \sigma_n^2)$ random variables. Suppose that
$$X_n \longrightarrow^D X$$
for some random variable X. Show that the sequence μ_n and σ_n^2 have limits $\mu \in \mathbb{R}$ and $\sigma^2 \geq 0$, and that X is $\mathcal{N}(\mu, \sigma^2)$.

   Hint: φ_{X_n} and φ_X being the characteristic functions of X_n and X, write
$$\varphi_{X_n} = e^{iu\mu_n - \frac{u^2\sigma_n^2}{2}}$$
and use Levy's Theorem to obtain that
$$\varphi_X = e^{iu\mu - \frac{u^2\sigma^2}{2}}$$
for some $\mu \in \mathbb{R}$ and $sigma^2 \geq 0$.

2. **(ex19.3 pp171).** Let $(X_n)_{n\geq 1}$, $(Y_n)_{n\geq 1}$ be sequences with X_n and Y_n defined on the same space for each n. Suppose
$$X_n \longrightarrow^D X \quad \text{and} \quad Y_n \longrightarrow^D Y,$$
and assume X_n and Y_n are independent for all n and that X and Y are independent. Show that
$$X_n + Y_n \longrightarrow^D X + Y.$$

3. **(ex20.7 pp178).** Let $(X_j)_{j\geq 1}$ be i.i.d. $\mathcal{N}(1, 3)$. Show that
$$\lim_{n\to\infty} \frac{X_1 + X_2 + \cdots + X_n}{X_1^2 + X_2^2 + \cdots + X_n^2} = \frac{1}{4} \ a.s.$$

4. **(ex21.4 pp186).** Let $(X_j)_{j\geq 1}$ be i.i.d. nonnegative with $E(X_1) = 1$ and $\sigma_{X_1}^2 = \sigma^2 \in (0, \infty)$. Show that

$$\frac{2}{\sigma}\left(\sqrt{S_n} - \sqrt{n}\right) \longrightarrow^{D} Z$$

with $\mathcal{L}(Z) = \mathcal{N}(0,1)$.

Hint: $\dfrac{S_n - n}{\sqrt{n}} = \dfrac{\sqrt{S_n} + \sqrt{n}}{\sqrt{n}}\left(\sqrt{S_n} - \sqrt{n}\right)$

5. **(ex21.11 pp187).** Let $(X_j)_{j\geq 1}$ be independent and let X_j have the uniform distribution on $(-j, j)$.

   (a) Show that

   $$\lim_{n\to\infty} \frac{S_n}{n^{3/2}} = Z$$

   in distribution where $\mathcal{L}(Z) = \mathcal{N}(0, 1/9)$.

   Hint: Show that the characteristic function of X_j is $\varphi_{X_i}(u) = \frac{\sin uj}{uj}$; compute $\varphi_{S_n}(u)$, then $\varphi_{S_n/n^{3/2}}(u)$, and prove that the limit is $e^{-u^2/18}$ by using $\sum_{j=1}^{n} = \frac{n(n+1)(2n+1)}{6}$.

   (b) Show that

   $$\lim_{n\to\infty} \frac{S_n}{\sqrt{\sum_{j=1}^{n} \sigma_j^2}} = Z$$

   in distribution where $\mathcal{L}(Z) = \mathcal{N}(0,1)$.

   Note: this is not a particular case of Theorem 21.2