

Home work # 6
Math 5290-Fall 2008
Due date 12/05/08 in class

Perform 2 out of 4 exercises!

1. **(ex1.5.2 pp28)**. Let $(X_n)_{n \geq 0}$ be a Markov chain in $\{0, 1, 2, \dots\}$ with transition probabilities given by

$$p_{01} = 1, \quad p_{i,i+1} + p_{i,i-1} = 1, \quad p_{i,i+1} = \left(\frac{i+1}{i}\right)^2 p_{i,i-1}, \quad i \geq 1.$$

Show that

$$P(X_n \rightarrow \infty \text{ as } n \rightarrow \infty) = 1.$$

2. **(ex1.5.3 pp28)**. Denote by T_j the first passage time to state j and set

$$f_{ij}^{(n)} = P_i(T_j = n).$$

- (a) Justify the identity

$$p_{ij}^{(n)} = \sum_{k=1}^n f_{ij}^{(k)} p_{jj}^{(n-k)} \text{ for } n \geq 1.$$

- (b) Deduce that

$$P_{ij}(s) = \delta_{ij} + F_{ij}(s)P_{jj}(s)$$

where

$$P_{ij}(s) = \sum_{n=0}^{\infty} p_{ij}^{(n)} s^n, \quad F_{ij}(s) = \sum_{n=0}^{\infty} f_{ij}^{(n)} s^n.$$

- (c) Show that $P_i(T_j < \infty) = 1$ if and only if $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$.

3. On the island Elschendieden there lives and almost extinct species called Vugiel. Vugiel's males can produce zero, one, two, ... male offspring with probability p_0, p_1, p_2, \dots respectively, where $p_i \geq 0$ and $\sum_{i=0}^{\infty} p_i = 1$. A challenging problem would be to find the Vugiel's chance of survival assuming that each individual lives exactly one year.

- (a) Write the problem in language of Markov chains.
(b) Suppose that

$$p_m = \frac{\lambda^m}{m!} e^{-\lambda}, \quad m \in \mathbb{N}.$$

Show that the transition probabilities are given by

$$p_{ij} = \frac{(\lambda i)^j}{j!} e^{-\lambda i}, \quad j, i \geq 0.$$

- (c) Show that the probability of survival is equal to 0 if $\lambda \leq 1$.
4. Consider a symmetric random walk on \mathbb{Z} . Show that 0 is null-recurrent state. Can you deduce whether other state are positive recurrent or null-recurrent?

Hint: Use ex2 and the infinite series.