Midterm
Math 5200-Spring 2009
March 12th 2009.

1. (i) Show that a countable union of sets of measure zero is again a set of measure zero.
   (ii)A set U is said to be a G_δ-set if U is the intersection of a countable collection of open sets. Prove that if A has measure zero then there exists a G_δ-set U of measure zero such that $A \subseteq U$.

2. Let f be a function with measurable domain D. Show that f is measurable if and only if the function g defined by $g(x) = f(x)$ for $x \in D$ and $g(x) = 0$ for $x \notin D$ is measurable.

3. Compute the following limit if it exists

$$\lim_{n\to\infty} \int_0^\infty n \sin \frac{x}{n} (x(1 + x^2))^{-1} dx$$

4. Show the following

   (a) Let f be integrable over $(-\infty, \infty)$, then

$$\int f(x)dx = \int f(x + t)dx, \ t \in (-\infty, \infty)$$

   (b) Let g be a bounded measurable function, then

$$\lim_{t\to 0} \int |g(x)\,[f(x) - f(x + t)]\,|dx = 0.$$

5. Prove the following variant of Fatou's Lemma: If $\{f_n\}$ is a sequence of non-negative measurable functions which converges to f a.e. and $\int_X f_n(x)dx \leq M < \infty$ for all n, then f is integrable and $\int_X f(x)dx \leq M$.