

Mathematics 4255

Midterm 1 with solutions

February 19, 2009

Partial credit will be awarded for your answers, so it is to your advantage to explain your reasoning and what theorems you are using when you write your solutions. Please answer the questions in the space provided and show your computations.

Good luck!

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Total	

Name: _____

I. **10points** Two dices are thrown. What is the probability that the total number of dots is

1. equal 7
2. an even number

Solutions: Let us denote by S the sample space which in this case is

$$S = \{(i, j), 1 \leq i \leq 6, 1 \leq j \leq 6\},$$

and the number of elements of S is 36.

1. Let E be the event that the total number of dots is 7,

$$E = \{(i, j), i + j = 7, 1 \leq i \leq 6, 1 \leq j \leq 6\},$$

and the number of elements of E is 6. Hence

$$P(E) = \frac{6}{36} = \frac{1}{6}.$$

2. Let us denote by F the event that the total number of dots is an even number

$$F = \{(i, j), i + j = \text{even}, 1 \leq i \leq 6, 1 \leq j \leq 6\},$$

the number of elements of F is 18. Hence

$$P(E) = \frac{18}{36} = \frac{1}{2}.$$

II. **10points** Let us denote by S a sample set and suppose that \mathcal{A} is the family of all subsets of S . We fix $x \in S$ and define a set-function

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

1. Show that δ_x is a probability measure.
2. What is the largest set of measure 0? What is the largest set of measure 1.

Solutions:

1. Since x is an arbitrary element of S $\delta_x(S) = 1$ and $\delta_x(\emptyset) = 0$.

Now let us take a sequence of events $\{A_i\}$ pairwise disjoint, $A_i \cap A_j = \emptyset$ Then either (a) $x \in \bigcup_i A_i$ or (b) $x \notin \bigcup_i A_i$.

In the case (b), $\delta_x(\bigcup_i A_i) = 0$. On the other side

$$\begin{aligned} x \notin \bigcup_i A_i &\implies x \notin A_i \forall i \\ &\implies \delta_x(A_i) = 0 \forall i \\ &\implies \sum_i \delta_x(A_i) = 0 \end{aligned}$$

In the case (a), $\delta_x(\bigcup_i A_i) = 1$. On the other side if $x \in \bigcup_i A_i$ then x belongs to at least one of the A_i 's. Suppose, it belongs to A_k then $\delta_x(A_k) = 1$ and if $k \neq i$ then $A_k \cap A_i = \emptyset$, hence x cannot belong to A_i and as a consequence $\delta_x(A_i) = 0$ for all $i \neq k$. Hence

$$\sum_i \delta_x(A_i) = \delta_x(A_k) = 1$$

Thus, in both cases we have that

$$\delta_x\left(\bigcup_i A_i\right) = \sum_i \delta_x(A_i).$$

2. The largest set of measure 0 is $S - \{x\}$ and the smallest set of measure 1 is $\{x\}$.

III. **10points** Groping about in the dark, we open one of the two drawers in a chest and pick up an item of clothing at random.

1. What is the probability that it is a sock if one drawer contains 6 socks and 6 underpants and the other contains 2 socks and 4 handkerchiefs.
2. What is the probability that the item comes from the first drawer if it turns out to be a sock.

Solutions: Let us denote by S the event that the item is a sock, by D_1 the event that the item comes from the first drawer and by D_2 the event that the item comes from the second drawer.

1. We are looking at the following probability $P(S)$. Using the partition equation we have that

$$P(S) = P(S|D_1)P(D_1) + P(S|D_2)P(D_2).$$

On the other side, we have that

$$P(D_1) = P(D_2) = \frac{1}{2},$$

and

$$P(S|D_1) = \frac{6}{12} = \frac{1}{2},$$

and

$$P(S|D_2) = \frac{2}{6} = \frac{1}{3}.$$

Hence,

$$P(S) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} = \frac{5}{12}.$$

2. Now we are looking at $P(D_1|S)$.

$$\begin{aligned} P(D_1|S) &= \frac{P(D_1 \cap S)}{P(S)} \\ &= \frac{P(S|D_1)P(D_1)}{P(S)} \\ &= \frac{\left(\frac{1}{2} \cdot \frac{1}{2}\right)}{\frac{5}{12}} = \frac{3}{5} \end{aligned}$$

VI. **10 points** Prove the following statements:

1. If $P(A) = 0$, then what is $P(A \cap B)$?
2. If $P(A) = 1$, then what is $P(A \cap B)$?

Solutions: Let us denote by S the sample space.

1. Since $A \cap B \subset A$ then $0 \leq P(A \cap B) \leq P(A) = 0$. Hence $P(A \cap B) = 0$.
2. $P(A) = 1$ implies that $P(A^c) = 1 - P(A) = 0$. Now,

$$\begin{aligned} P(B) &= P(B \cup S) \\ &= P(B \cup (A \cap A^c)) \\ &= P((B \cap A) \cup (B \cap A^c)) \\ &= P(B \cap A) + P(B \cap A^c) \\ &= P(B \cap A). \end{aligned}$$

We have used the fact that $B \cap A^c \subset A^c$ hence

$$0 \leq P(B \cap A^c) \leq P(A^c) = 0 \implies P(B \cap A^c) = 0.$$

V. **10 points** Independent flips of a coin that lands on heads with probability p are made. What is the probability that the first four outcomes are

1. H, H, H, H.
2. T, H, H, H.

Solutions: Let us denote by H_i the event that head occurs at the i th flip. $P(H_i) = p$.

- 1.

$$P(H, H, H, H) = P(H_1 \cap H_2 \cap H_3 \cap H_4) = P(H_1)P(H_2)P(H_3)P(H_4) = p^4.$$

- 2.

$$P(T, H, H, H) = P(H_1^c \cap H_2 \cap H_3 \cap H_4) = P(H_1^c)P(H_2)P(H_3)P(H_4) = (1 - p)p^3.$$

VI. (**Bonus 10 points**) Let $S = \{1, 2, \dots, n\}$ and suppose that A and B are, independently, equally likely to be any of the 2^n subsets (including the null set and S itself) of S .

1. Show that

$$P(A \subset B) = \left(\frac{3}{4}\right)^n.$$

Hint: Let $N(B)$ denote the number of elements in B . Use

$$P(A \subset B) = \sum_{i=0}^n P(A \subset B | N(B) = i) P(N(B) = i).$$

2. Show that

$$P(A \cap B = \emptyset) = \left(\frac{3}{4}\right)^n.$$