Mathematics 4255
Midterm 2 with solutions

April 2nd, 2009

Partial credit will be awarded for your answers, so it is to your advantage to explain your reasoning and what theorems you are using when you write your solutions. Please answer the questions in the space provided and show your computations.

Good luck!

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Name:____________________________________

I. **(10points)** The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} x+y & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & otherwise \end{cases}$$

1. Are X and Y independent?
2. Find $P(X+Y<1)$.

Solution:

1. For $0 < x < 1$

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy = \int_0^1 (x+y)dy \\ &= (xy + y^2/2)|_0^1 = x + 1/2. \end{aligned}$$

And for $0 < y < 1$

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)dx = \int_0^1 (x+y)dx \\ &= (x^2/2 + xy)|_0^1 = y + 1/2. \end{aligned}$$

Hence

$$f_X(x) \cdot f_Y(y) = (x+1/2)(y+1/2) \neq (x+y) = f(x,y).$$

Thus, X and Y are not independent.

2. Let

$$C = \left\{(x,y) \in (0,1)^2,\ x+y<1\right\} = \{(x,y),\ 0<x<1,\ 0<y<1-x\}.$$

$$\begin{aligned} P\{X+Y<1\} &= P\{(X,Y)\in C\} = \int_0^1 \int_0^{1-x} (x+y)dydx \\ &= \int_0^1 [x(1-x) + (1-x)^2/2]dx = 1/3. \end{aligned}$$

II. **(10points)** Assume X and Y are independent exponential random variables with common parameter λ.

1. Find the probability density function of $Z = X + Y$.
2. Find $P(Y < t | X + Y > s)$, for some $s > 0$ and $t > 0$.

Hint: Recall that X is an exponential random variable with parameter λ if its probability density function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Solution:

1.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & otherwise \end{cases}$$

and

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda y} & y \geq 0 \\ 0 & otherwise \end{cases}$$

Since X are Y are independent, the joint density function of X and Y is given by their product

$$f_{X,Y}(x,y) = \begin{cases} \lambda^2 e^{-\lambda(x+y)} & x \geq 0, \ y \geq 0 \\ 0 & otherwise \end{cases}$$

Let

$$\begin{aligned} C &= \{(x,y), \ x \geq 0, \ y \geq 0, \ x + y \leq a\} \\ &= \{(x,y), \ 0 \leq x \leq a - y, \ 0 \leq y \leq a\}. \end{aligned}$$

For $a > 0$

$$\begin{aligned} F_{X+Y}(a) &= P(X + Y \leq a) = P[(X,Y) \in C] \\ &= \int\int_C f_{X,Y}(x,y)dxdy = \lambda^2 \int\int_C e^{-\lambda(x+y)}dxdy \\ &= \lambda^2 \int_0^a \int_0^{a-y} e^{-\lambda x} e^{-\lambda y} dxdy = 1 - e^{-\lambda a}(1 + \lambda a). \end{aligned}$$

Hence $a > 0$

$$\begin{aligned} f_{X+Y}(a) &= \frac{d}{da} F_{X+Y}(a) \\ &= \frac{d}{da}\left(1 - e^{-\lambda a}(1 + \lambda a)\right) \\ &= \lambda^2 a e^{-\lambda a} \end{aligned}$$

Hence

$$f_{X+Y}(a) = \begin{cases} \lambda^2 a e^{-\lambda a} & a \geq 0 \\ 0 & otherwise \end{cases}$$

2.

$$\begin{aligned} P\left(Y < t | X + Y > s\right) &= \frac{P\left(Y < t, X + Y > s\right)}{P\left(X + Y > s\right)} \\ &= \frac{P[(X,Y) \in C_1]}{P[(X,Y) \in C_2]} \end{aligned}$$

Where

$$\begin{aligned} C_1 &= \{(x,y),\ x \geq 0,\ y \geq 0,\ y < t,\ x + y > s\} \\ &= \{(x,y),\ 0 < y < t,\ x > s - y\}. \end{aligned}$$

$$\begin{aligned} C_2 &= \{(x,y),\ x \geq 0,\ y \geq 0,\ x + y > s\} \\ &= \{(x,y),\ 0 < y,\ x > s - y\}. \end{aligned}$$

$$\begin{aligned} P[(X,Y) \in C_1] &= \lambda^2 \int_0^t \int_{s-y}^{\infty} e^{-\lambda(x+y)} dx dy \\ &= \lambda t e^{-\lambda s}. \end{aligned}$$

and

$$\begin{aligned} P[(X,Y) \in C_2] &= \lambda^2 \int_0^{\infty} \int_{s-y}^{\infty} e^{-\lambda(x+y)} dx dy \\ &= \lambda. \end{aligned}$$

Hence

$$P\left(Y < t | X + Y > s\right) = t e^{-\lambda s}.$$

III. **(10points)** Let X be a normal random variable with mean 12 and variance 4. Find the value of c such that $P(X > c) = 0.10$.

Solution:

By assumption, $X \sim N(12, 4)$. Let $Z := \frac{X-12}{2}$ then Z is a standard normal random variable.

$$\begin{aligned}
0.10 &= P(X > c) \\
&= P\left(\frac{X-12}{2} > \frac{c-12}{2}\right) \\
&= 1 - P\left(\frac{X-12}{2} \leq \frac{c-12}{2}\right) \\
&= 1 - P\left(Z \leq \frac{c-12}{2}\right) \\
&= 1 - \phi\left(\frac{c-12}{2}\right) \\
&\Longrightarrow \\
&\ \phi\left(\frac{c-12}{2}\right) = 0.90.
\end{aligned}$$

Looking at the table we get that

$$\frac{c-12}{2} \simeq 1.28 \Longrightarrow c \simeq 14.56.$$

VI. **(10 points)** The joint probability mass function of the random variable X, Y, Z is given by

$$p(1,2,3) = p(2,2,1) = p(2,3,2) = \frac{1}{3}.$$

Find $E[XYZ]$.

Solution:

$$\begin{aligned} E[XYZ] &= \sum_{x,y,z} xyzp(x,y,z) \\ &= \frac{1}{3}(1 \cdot 2 \cdot 3 + 2 \cdot 2 \cdot 1 + 2 \cdot 3 \cdot 2) \\ &= \frac{1}{3}(6 + 4 + 12) = 22/3. \end{aligned}$$

V. **(10 points)**

1. Let X be uniformly distributed over $(0,1)$. Let $Y := X^n$, $n \geq 1$ is an integer. Find the Probability density function (pdf) of the r.v Y.

2. Let X be a continuous random variable with probability density function f_X. Let $Y := X^2$. Find the probability density function of the r.v. Y.

Solution:

1. By assumption $X \sim U(0,1)$ and $Y := X^n$. Hence for $0 \leq y \leq 1$,

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(X^n \leq y) \\ &= P\left(X \leq y^{1/n}\right) \\ &= F_X\left(y^{1/n}\right), \end{aligned}$$

hence the pdf is given by

$$\begin{aligned} f_Y(y) &= \frac{d}{dy}\left(F_Y(y)\right) = \frac{d}{dy}\left(F_X\left(y^{1/n}\right)\right) \\ &= \frac{1}{n} f_X\left(y^{1/n}\right)\left(y^{1/n-1}\right). \end{aligned}$$

Hence

$$f_Y(y) = \begin{cases} \frac{1}{n} y^{1/n-1} & 0 \leq y \leq 1 \\ 0 & otherwise \end{cases}$$

2. By assumption, X is a continuous random variable with density f_X, then $y \geq 0$,

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P\left(X^2 \leq y\right) \\ &= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}). \end{aligned}$$

Hence, diffentiating in y, we get that

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left(f_X(\sqrt{y}) - f_X(-\sqrt{y})\right).$$

Thus,

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}}\left(f_X(\sqrt{y}) - f_X(-\sqrt{y})\right) & y \geq 0 \\ 0 & otherwise \end{cases}$$