Some rigorous results on a stochastic GOY model

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Abstract

A stochastic infinite dimensional version of the GOY model is rigorously investigated. Well posedness of strong solutions, existence and p-integrability of invariant measures is proved. Existence of solutions to the zero viscosity equation is also proved. With these preliminary results, the asymptotic exponents ζ_p of the structure function are investigated. Necessary and sufficient conditions for $\zeta_2 \geq 2/3$ and $\zeta_2 = 2/3$ are given and discussed on the basis of numerical simulations.

1 Introduction

The GOY model, from E. B. Gledzer, K. Ohkitani, M. Yamada, is a simplified Fourier system with respect to The Navier-Stokes one, where the interaction between different modes is preserved only between neighbors (see section 1.1). Questions related to the cascade of energy in turbulent flows from large to small scales could potentially be better understood in such a case.

There is an extended literature on the GOY model and, more generally, on shell models, however it is exclusively dedicated to the numerical approach and as a consequence, to the finite dimensional case (see, for instance, [2, 3,

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7, 16]). In a recent work [5] some results of regularity, attractors and inertial manifold are proved for a *deterministic* infinite dimensional shell model.

In the study of 3D turbulence, among the quantities of major interest are the asymptotic exponents ζ_p of the p-order structure function, defined in section 4.1. Numerical investigations as well as heuristics based on physical intuition have been extensively developed to support a multifractal structure of ζ_p (see for instance [2], [3], [16], [14] and references therein). There is general agreement on $\zeta_p < \frac{p}{3}$ for large p and on $\zeta_3 = 1$. Less clear is the value of ζ_2, prescribed to be $2/3$ by Kolmogorov theory [17] for 3D turbulent fluids; certain simulations and methods of fit gave a value larger than $2/3$. In section 4.4 we try to motivate the great dynamical interest in the difference between the two cases $\zeta_2 > 2/3$ and $\zeta_2 = 2/3$. Let us emphasize the lack of mathematically rigorous results on these questions.

On the basis of previous works, especially [20] and [12], we believe that some *rigorous* informations on questions of turbulence theory could be obtained from *stochastic* versions of the equations of fluid dynamics. In a sentence, the advantage of the stochastic case is the major simplicity of balance laws between mean rates of energy injection, dissipation and flux (and variants like mean rate of vortex stretching, see [12]), see proposition 14. The mean values are computed for a stationary state. Such a stationary state is not an equilibrium state and Gibbs paradigm does not apply. Thus there is at present no chance to compute mean rates from a statistical ensemble; the only hope is to write powerful relations between different rates, and this is accomplished by Itô formula and stochastic analysis.

Our aim in this research project on the stochastic GOY equations has thus been to investigate such balance laws in one of the simplest settings related to fluid dynamics.

With these general motivations in mind, we present here some preliminary results, although incomplete, with the hope to motivate further research. We first prove a number of necessary foundational results of well posedness and invariant measures. Precisely, in section 2 we prove existence of strong (in the probabilistic sense) solutions by a pathwise method, we prove pathwise uniqueness and some continuous dependence results, and show p-integrability of solutions under proper assumptions. As a side result, we consider the zero viscosity case and prove existence of global solutions. This is remarkable in constrast to the case of 2 and 3 dimensional Navier-Stokes equations, as we remark in section 2.4; our proof makes essential use of the finite size interaction of the non-linear term.

In section 3 we continue the preliminary part and prove the existence of invariant measures, their p-integrability necessary to introduce the structure function, and the basic balance relations between mean rates of dissipation, flow and energy injection, used in the second part of the paper.

Based on these rigorous foundations, in sections 4.1 and 4.2 we settle a framework to study asymptotic exponents, with some definitions and general elementary results, not specific of the GOY model. Finally, in section 5 we show how the balance laws of the stochastic GOY equations may add rigorous informations on the asymptotic exponents. Our main results concern the characterization of both the sentences $\zeta_2^- \geq 2/3$ and $\zeta_2 = 2/3$ in terms of upper and lower bounds on the ratio

$$\frac{E^\nu\left[|u_n|^2\right]}{\left|E^\nu\left[u_n u_{n+1} u_{n+2}\right]\right|^{2/3}}. \tag{1}$$

These results do not prove $\zeta_2^- \geq 2/3$ or $\zeta_2 = 2/3$ but give necessary and sufficient conditions for them. At present, the interest in these conditions is numerical: it offers an alternative way to explore the value of ζ_2 with respect to estimating a slope of a rather oscillating curve, which could be the origin of little discrepancies in the results. For the future, we may hope to understand something about the statistical dependence of the variables u_n, u_{n+1}, u_{n+2} and deduce theoretical informations on the ratio (1).

1.1 The model

The stochastic infinite dimensional GOY model is described by an infinite sequence $(u_n(t))_{n\geq -1}$ of complex valued functions ($u_n(t) = u_{n,1}(t) + iu_{n,2}(t)$) subject to the constraints

$$u_{-1}(t) = u_0(t) = 0$$

and to the equations for $n = 1, 2, ...$

$$\begin{aligned} & du_n + \nu k_n^2 u_n dt \\ & + ik_n \left(\frac{1}{4} \overline{u}_{n-1} \overline{u}_{n+1} - \overline{u}_{n+1} \overline{u}_{n+2} + \frac{1}{8} \overline{u}_{n-1} \overline{u}_{n-2} \right) dt \\ & = \sigma_n d\beta_n \end{aligned} \tag{2}$$

where $\overline{u}_n$ denotes the complex conjugate of u_n, $\nu > 0$ will be called viscosity, $k_n = 2^n k_0$ will be called wave numbers ($k_0 > 0$ given). Heuristically, in the GOY model, the variable u_n will correspond to an averege value of the Navier-Stokes Fourier components of wavenumbers in a "shell", namely the interval $k_0(2^n,\ 2^{n+1})$. The reason for the exponential growth of the the size of the shells is to mimic Kolmogorov cascade (see for instance [16, **?**, 7]).
$(\sigma_n)_{n\geq 1}$ are 2×2 real matrices, the "intensities" of the noise, and $(\beta_n)_{n\geq 1}$ is a sequence of independent complex-values Brownian motions on a probability space $(\Omega, \mathcal{F}, P)$, with expectation denoted by E. The assumptions on σ_n are very general in the foundational part of the paper, but the case of main interest in view of K41 theory is when σ_n is different from zero only for the first few values of n (noise acting only on the largest scales). We could extend many theoretical results of this paper to the case of u-dependent coefficients $(\sigma_n(u))_{n\geq 1}$, under appropriate assumptions, but this generality is at present poorly motivated, so we restrict to a basic case.

1.2 Infinite dimensional set-up

Similarly to the theory of Navier-Stokes equations, we would like to rewrite system (2) as an infinite dimensional equation of the form (3) below. To this aim, let us introduce some function spaces and operators.

Let us introduce the following spaces of complex valued sequences; we consider them as vector spaces on the field of real numbers. The space

$$H = \left\{ u = (u_1, ...) \in \mathbb{C}^\infty : \sum_{n=1}^{\infty} |u_n|^2 < \infty \right\}$$

is a (real) Hilbert space with the inner product

$$\langle u, v \rangle_H := \mathrm{Re} \sum_{n=1}^{\infty} u_n \overline{v}_n$$

and the norm given by $|u|_H^2 = \sum_{n=1}^{\infty} |u_n|^2$. Let us recall that we have defined $k_n = 2^n k_0$, $n \geq 1$, with $k_0 > 0$ given. We introduce now the Hilbert spaces $D(A) \subset V \subset H$ defined as

$$V = \left\{ u \in H : \sum_{n=1}^{\infty} k_n^2 |u_n|^2 < \infty \right\}$$

with norm $\|u\|_V^2 = \sum_{n=1}^{\infty} k_n^2 |u_n|^2$, and

$$D(A) = \left\{ u \in H : \sum_{n=1}^{\infty} k_n^4 |u_n|^2 < \infty \right\}.$$

On the latter space we define the linear operator $A : D(A) \subset H \to H$ as

$$(Au)_n = k_n^2 u_n, \quad \forall u \in D(A).$$

The operator A is selfadjoint and strictly positive definite:

$$\langle Au, u \rangle_H \geq k_0 |u|_H^2, \quad \forall u \in D(A).$$

We finally introduce the bilinear operator $B(.,.) : V \times H \to H$ defined as

$$B(u,v)_n = ik_n \left(\frac{1}{4} \overline{v}_{n-1} \overline{u}_{n+1} - \frac{1}{2} (\overline{u}_{n+1} \overline{v}_{n+2} + \overline{v}_{n+1} \overline{u}_{n+2}) + \frac{1}{8} \overline{u}_{n-1} \overline{v}_{n-2} \right).$$

Since

$$\sum_{n=1}^{\infty} k_n^2 |u_n|^2 |v_n|^2 \leq \left(\sup_n k_n^2 |u_n|^2 \right) \sum_{n=1}^{\infty} |v_n|^2 \leq \|u\|_V^2 |v|_H^2$$

it is easy to verify that $B(u,v) \in H$ when $(u,v) \in V \times H$; but also when $(u,v) \in H \times V$, so we may define B also in the spaces $B(.,.) : H \times V \to H$. Let us summarize this fact:

Lemma 1 *There is a constant $C > 0$ such that*

$$|B(u,v)|_H \leq C \|u\|_V |v|_H$$

and

$$|B(u,v)|_H \leq C \|v\|_V |u|_H$$

for u and v in the proper spaces.

We have

$$\langle B(u,v), v \rangle = 0$$

whenever defined. Indeed,

$$
\begin{aligned}
&\frac{-i}{k_0}\left\langle B(u,v),v\right\rangle\\
=&\frac{-i}{k_0}\sum_{n=1}^{\infty}ik_n\left(2^{-2}\overline{v}_{n-1}\overline{u}_{n+1}-2^{-1}\left(\overline{u}_{n+1}\overline{v}_{n+2}+\overline{v}_{n+1}\overline{u}_{n+2}\right)+2^{-3}\overline{u}_{n-1}\overline{v}_{n-2}\right)\overline{v}_n\\
&=\sum_{n=1}^{\infty}2^{n-2}\overline{v}_{n-1}\overline{v}_n\overline{u}_{n+1}-\sum_{n=1}^{\infty}2^{n-1}\overline{v}_n\overline{u}_{n+1}\overline{v}_{n+2}\\
&-\sum_{n=1}^{\infty}2^{n-1}\overline{v}_n\overline{v}_{n+1}\overline{u}_{n+2}+\sum_{n=1}^{\infty}2^{n-3}\overline{v}_{n-2}\overline{u}_{n-1}\overline{v}_n\\
&=\sum_{n=0}^{\infty}2^{n-1}\overline{v}_n\overline{v}_{n+1}\overline{u}_{n+2}-\sum_{n=1}^{\infty}2^{n-1}\overline{v}_n\overline{u}_{n+1}\overline{v}_{n+2}\\
&-\sum_{n=1}^{\infty}2^{n-1}\overline{v}_n\overline{v}_{n+1}\overline{u}_{n+2}+\sum_{n=-1}^{\infty}2^{n-1}\overline{v}_n\overline{u}_{n+1}\overline{v}_{n+2}
\end{aligned}
$$

by change of variable, and this is zero.

Remark 2 *There are infinitely many operators $B(u,v)$ with the previous properties which extend $B(u,v)$. The present choice looks more elegant.*

Consider also the space V' defined as

$$V'=\left\{u=(u_1,...)\in\mathbb{C}^\infty:\sum_{n=1}^{\infty}k_n^{-2}\left|u_n\right|^2<\infty\right\}.$$

We have $H\subset V'$ and V' is the dual of V (with respect to H), with dual pairing between V' and V defined as

$$\left\langle u,v\right\rangle_{V',V}:=\operatorname{Re}\sum_{n=1}^{\infty}u_n\overline{v}_n,\quad\forall u\in V',v\in V.$$

It coincides with $\left\langle u,v\right\rangle_H$ when $u\in H$.

It is easy to extend A as a bounded linear operator from V to V'. One can also extend B to a bilinear operator $B\left(.,.\right):H\times H\to V'$. The definition is possible because

$$|B(u,v)|_{V'}^2=\sum_{n=1}^{\infty}k_n^{-2}\left|B(u,v)_n\right|^2\leq C\sum_{n=1}^{\infty}\bar{v}_n^2\bar{u}_n^2\leq C|u|_H|v|_H.$$

We still have

$$\langle B\left(u,v\right),z\rangle_{V',V}=-\left\langle B\left(u,z\right),v\right\rangle_H$$

with now $u,v\in H$, $z\in V$.

2 Well posedness

Existence of solutions can be proved in several ways. First, one can follow the pathwise approach described here as well as an approach based on solutions to the martingale problem. The one chosen here seems to be more elementary. Second, one can follow general lines of proof inspired to the theory of Navier-Stokes equations or one can use particular tricks related to the finite-range interaction of the modes. We mostly adopt the general viewpoint which is more unifying, but we strongly use the finite-range interaction in the section on the $\nu\to 0$ limit.

Let us start with pathwise estimates on the difference of two solutions. Even if the model resembles the 3D Navier-Stokes equations for some qualitative aspects, these estimates are even stronger than those of the 2D case.

2.1 Pathwise uniqueness and continuous dependence on initial conditions

Consider the equation

$$du\left(t\right)=\left[-\nu Au\left(t\right)-B\left(u\left(t\right),u\left(t\right)\right)\right]dt+dW\left(t\right),\quad t\geq 0 \tag{3}$$

where $\left(W\left(t\right)\right)_{t\geq 0}$ is a Brownian motion in H defined on a filtered probability space $\left(\Omega,\mathcal{F},\left(\mathcal{F}_t\right)_{t\geq 0},P\right)$, with nuclear covariance operator Q (a nuclear semi-definite symmetric operator in H). We impose an initial condition given by an $\mathcal{F}_0$-measurable random variable $u_0:\Omega\to H$. As usual, we interpret the equation in the integral weak sense

$$\begin{aligned}&\langle u(t),\varphi\rangle_H+\int_0^t\nu\left\langle u(s),A\varphi\right\rangle_H ds-\int_0^t\left\langle B\left(u(s),\varphi\right),u(s)\right\rangle_H ds\\&=\left\langle u_0,\varphi\right\rangle_H+\left\langle W(t),\varphi\right\rangle_H\end{aligned} \tag{4}$$

with $\varphi \in D(A)$. When $u(s) \in V$, we have

$$-\langle B(u(s), \varphi), u(s)\rangle_H = \langle B(u(s), u(s)), \varphi\rangle_H$$

while in general if $u(s) \in H$, we have

$$-\langle B(u(s), \varphi), u(s)\rangle_H = \langle B(u(s), u(s)), \varphi\rangle_{V',V}$$

so (4) is a generalized version of (3), with a meaning even if just $u(s) \in H$ (at least integrable in s).

Definition 3 *We say that a continuous adapted processes in H is a solutions of (3) if P-a.s. the integral equation (4) is satisfied for every $t \geq 0$ and $\varphi \in D(A)$.*

Theorem 4 *Let $\left(u^{(1)}(t)\right)_{t\geq 0}$, $\left(u^{(2)}(t)\right)_{t\geq 0}$, be two continuous adapted solutions of (3) in H, with initial conditions $u_0^{(1)}$ and $u_0^{(2)}$ as above. Then there is a constant $C_\nu > 0$, depending only on ν, such that*

$$\begin{aligned}
&\left|u^{(1)}(t) - u^{(2)}(t)\right|_H^2 \\
&\leq e^{C_\nu\left(\int_0^t\left|u^{(1)}(s)\right|_H^2 + \int_0^t\left|u^{(2)}(s)\right|_H^2\right)ds}\left|u_0^{(1)} - u_0^{(2)}\right|_H^2, \quad t \geq 0
\end{aligned}$$

with probability one. In particular, if $u_0^{(1)} = u_0^{(2)}$, then

$$P\left(u^{(1)}(t) = u^{(2)}(t) \text{ for every } t \geq 0\right) = 1.$$

Proof. For $n \in \mathbb{N}$, let $J_n : H \to D(A)$ be the Yosida approximations defined as $J_n = n(n+A)^{-1}$ (in the following proof one can replace J_n by π_n defined in the sequel; we use J_n to explore one more tool). It is easy to check or well known that J_n are selfadjoint in H, commute with A, $\lim_{n\to\infty} J_n x = x$ for every $x \in H$, the extensions $J_n : V' \to V$ are well defined and equibounded and $\lim_{n\to\infty} J_n x = x$ for every $x \in V'$. Let $u(t)$ be a solution and

$$u_n(t) = J_n u(t).$$

From (4) with $\varphi = J_n\psi$ and from the extension results of the previous section we have

$$\begin{aligned}
&\langle u_n(t), \psi\rangle_H + \int_0^t \nu\langle Au_n(s), \psi\rangle_H ds + \int_0^t \langle J_n B(u(s), u(s)), \psi\rangle_H ds \\
&= \langle u_n(0), \psi\rangle_H + \langle J_n W_t, \psi\rangle_H .
\end{aligned}$$

So we may write the integral equation in H

$$u_n(t) + \int_0^t \nu A u_n(s) ds + \int_0^t J_n B\left(u(s), u(s)\right) ds = u_n(0) + J_n W_t.$$

With the notation $v(t) = u^{(1)}(t) - u^{(2)}(t)$, we have

$$v_n(t) + \int_0^t \nu A v_n(s) ds = v_n(0) - \int_0^t J_n B\left(u^{(1)}(s), v(s)\right) ds$$
$$- \int_0^t J_n B\left(v(s), u^{(2)}(s)\right) ds.$$

This implies that $v_n(t)$ is differentiable in t and by the chain rule in H we have

$$\frac{1}{2}\frac{d}{dt} |v_n(t)|_H^2 + \nu \|v_n(t)\|_V^2 = - \left\langle J_n B\left(u^{(1)}(t), v(t)\right), v_n(t)\right\rangle_H$$
$$- \left\langle J_n B\left(v(t), u^{(2)}(t)\right), v_n(t)\right\rangle_H.$$

Now

$$\begin{aligned}\left|\left\langle J_n B\left(u^{(1)}(t), v(t)\right), v_n(t)\right\rangle_H\right| &\leq C \left|J_n B\left(u^{(1)}(t), v(t)\right)\right|_{V'} \|v_n(t)\|_V \\ &\leq C \left|u^{(1)}(t)\right|_H |v(t)|_H \|v_n(t)\|_V \\ &\leq \frac{\nu}{4} \|v_n(t)\|_V^2 + C_\nu \left|u^{(1)}(t)\right|_H^2 |v(t)|_H^2\end{aligned}$$

and, in the sane way,

$$\left|\left\langle J_n B\left(v(t), u^{(2)}(t)\right), v_n(t)\right\rangle_H\right| \leq \frac{\nu}{4} \|v_n(t)\|_V^2 + C_\nu \left|u^{(2)}(t)\right|_H^2 |v(t)|_H^2.$$

Therefore

$$|v_n(t)|_H^2 \leq |v_n(0)|_H^2 + C_\nu \int_0^t \left(\left|u^{(1)}(s)\right|_H^2 + \left|u^{(2)}(s)\right|_H^2\right) |v(s)|_H^2 ds.$$

As $n \to \infty$ we get

$$|v(t)|_H^2 \leq |v(0)|_H^2 + C_\nu \int_0^t \left(\left|u^{(1)}(s)\right|_H^2 + \left|u^{(2)}(s)\right|_H^2\right) |v(s)|_H^2 ds.$$

By Gronwall lemma, the proof is completed. ■

2.2 Existence: pathwise solution

Given

$$u_0 \in H, \quad \omega \in C^{\alpha}\left([0,T];H\right)$$

for some $\alpha > 0$, we consider the *deterministic* equation

$$\begin{aligned} &\langle u(t), \varphi\rangle_H + \int_0^t \nu \langle u(s), A\varphi\rangle_H \, ds - \int_0^t \langle B\left(u(s), \varphi\right), u(s)\rangle_H \, ds \\ &= \langle u_0, \varphi\rangle_H + \langle \omega(t), \varphi\rangle_H \end{aligned} \tag{5}$$

with $\varphi \in D(A)$. It can be differentiated in time only in the sense of distributions. Our aim is to prove the existence of a solution u, continuous in H. The uniqueness is true as in the previous section.

We introduce the finite dimensional subspaces H_n of H given by all $u \in H$ with zero components except for $u_1, ..., u_n$:

$$H_n = \{u \in H : u_j = 0 \ \forall j > n\}.$$

Then we introduce the finite dimensional orthogonal projections $\pi_n : H \to H_n$ and consider the stochastic differential equation in H_n

$$u^{(n)}(t) + \int_0^t \left[\nu A u^{(n)}(s) + \pi_n B\left(u^{(n)}(s), u^{(n)}(s)\right)\right] ds = \pi_n u_0 + \pi_n \omega(t). \tag{6}$$

By the classical contraction principle, it is very easy to prove existence and uniqueness of a solution $\left(u^{(n)}(t)\right)_{t\in[0,\tau]}$ on some time interval $[0,\tau]$ (local solution). Alternatively, one may apply the well known Cauchy theorem for differential equations of the form $y' = F(t,y)$ with F continuous in (t,y), locally Lipschitz continuous in y uniformly in t, to the equation for $v^{(n)}(t)$ below.

We are going to prove now an a priori estimate that implies that the solution is global in time. Consider the auxiliary linear equation in H_n

$$z^{(n)}(t) + \int_0^t \nu A z^{(n)}(s)\, ds = \pi_n \omega(t). \tag{7}$$

Lemma 5 *There exists a unique global continuous solution $z^{(n)}$ in H_n, given by*

$$z^{(n)}(t) = \pi_n z(t)$$

where $z \in C([0,T];H)$ is given by

$$z(t) = S(t)\omega(t) - \int_0^t \nu AS(t-s)(\omega(s) - \omega(t))\, ds \tag{8}$$

where $S(t)$ is the analytic semigroup in H generated by νA.

The proof of this lemma is based on standard techniques that can be found, for instance, in [10].

We state the following preliminary result.

Theorem 6 *Equation (6) has a unique continuous solution $\left(u^{(n)}(t)\right)_{t\geq 0}$ in H. In addition there is a constant $C\left(T, |u_0|_H, |\omega|_{C^\alpha([0,T];H)}\right)$, independent of n, such that (11) holds true.*

Proof. We introduce the function $v^{(n)}(t) := u^{(n)}(t) - z^{(n)}(t)$, $t \in [0,\tau]$, which is differentiable on $[0,\tau]$ and satisfies

$$\frac{dv^{(n)}}{dt} + \nu Av^{(n)} + \pi_n B\left(u^{(n)}, u^{(n)}\right) = 0 \tag{9}$$

with the initial condition $v^{(n)}(0) = \pi_n u_0$. Thus, using the same arguments of the proof of theorem 4 we get the estimate

$$\frac{1}{2}\frac{d}{dt}\left|v^{(n)}\right|_H^2 + \frac{\nu}{2}\left\|v^{(n)}\right\|_V^2 \leq C_\nu\left(\left|z^{(n)}\right|_H^2\left|v^{(n)}\right|_H^2 + \left|z^{(n)}\right|_H^4\right). \tag{10}$$

By Gronwall lemma

$$\left|v^{(n)}(t)\right|_H^2 \leq \left|v^{(n)}(0)\right|_H^2 e^{\int_0^t C\left|z^{(n)}(s)\right|_H^2 ds} + \int_0^t e^{\int_\sigma^t C\left|z^{(n)}(s)\right|_H^2 ds} C\left|z^{(n)}(\sigma)\right|_H^4 d\sigma.$$

Since $z^{(n)}$ is continuous in H, with $\sup_{t\in[0,T]}\left|z^{(n)}(t)\right|_H^2$ depending only on the norm of ω in $C^\alpha([0,T];H)$ (with constants independent of n) and $v^{(n)}(0) = \pi_n u_0$, we deduce that

$$\sup_{t\in[0,\tau]}\left|v^{(n)}(t)\right|_H^2 \leq C\left(T, |u_0|_H, |\omega|_{C^\alpha([0,T];H)}\right).$$

Therefore

$$\sup_{t\in[0,\tau]}\left|u^{(n)}(t)\right|_H^2 \leq C\left(T, |u_0|_H, |\omega|_{C^\alpha([0,T];H)}\right). \tag{11}$$

This a priori bound give us the global existence of $u^{(n)}$ on $[0,T]$. The proof is complete. ■

We can now prove the following result.

Theorem 7 *Given*

$$u_0 \in H, \quad \omega \in C^{\alpha}([0,T];H)$$

for some $\alpha > 0$, there exists one and only one solution u in $C([0,T];H)$ of the deterministic equation (5).

Proof. From (10) and the previous estimates, we have

$$\int_0^T \left\| v^{(n)}(t) \right\|_V^2 dt \le C\left(\nu, T, |u_0|_H, |\omega|_{C^{\alpha}([0,T];H)}\right)$$

Moreover, from (9) we have

$$\left| \frac{dv^{(n)}(t)}{dt} \right|_{V'} \le \nu C_A \left\| v^{(n)}(t) \right\|_V + C \left| u^{(n)}(t) \right|_H^2$$

and thus

$$\left| v^{(n)} \right|_{W^{1,2}(0,T;V')} \le C\left(\nu, T, |u_0|_H, |\omega|_{C^{\alpha}([0,T];H)}\right)$$

with a new constant independent of n.

We have proved that the sequence $\left\{v^{(n)}\right\}$ is bounded in $L^2(0,T;V)$ and $W^{1,2}(0,T;V')$. Hence, by a classical compactness theorem [21], there is a subsequence $\left\{v^{(n_k)}\right\}$ which converges strongly to some v in $L^2(0,T;H)$ and $C([0,T],D(A)')$. By difference, $\left\{u^{(n_k)}\right\}$ converges strongly to $u = v + z$ in the same topologies. This convergence allows us to pass to the limit from (6) to (5) for every $\varphi \in D(A)$.

Finally, the boundedness of $\left\{v^{(n)}\right\}$ in $L^2(0,T;V)$ and $W^{1,2}(0,T;V')$ implies that v is in $L^2(0,T;V)$ and $W^{1,2}(0,T;V')$, hence in $C([0,T];H)$ by a well known theorem, [22]. Thus also $u \in C([0,T];H)$. The proof is complete ■

2.3 Continuous dependence on ω, progressive measurability, solution to equation (3)

With methods similar to those of uniqueness we may prove that the solution of the deterministic equation (5) depends continuously on ω.

Theorem 8 *Given*

$$u_0 \in H, \quad \omega^{(1)}, \omega^{(2)} \in C^{\alpha}([0,T];H)$$

for some $\alpha > 0$, the solutions $u^{(1)}$, $u^{(2)}$ corresponding to $\omega^{(1)}$, $\omega^{(2)}$ satisfy

$$\sup_{t\in[0,T]} \left|u^{(1)}(t) - u^{(2)}(t)\right|_H$$

$$\leq C\left(\nu, T, |u_0|_H, \int_0^T \left|u^{(1)}(r)\right|_H^2 dr, \int_0^T \left|u^{(2)}(r)\right|_H^2 dr\right) \left|\omega^{(1)} - \omega^{(2)}\right|_{C^\alpha([0,T];H)}.$$

Proof. Denote by $z^{(1)}$, $z^{(2)}$ the functions (8) with respect to $\omega^{(1)}$, $\omega^{(2)}$. Let

$$y = v^{(1)} - v^{(2)} = u^{(1)} - u^{(2)} - \left(z^{(1)} - z^{(2)}\right).$$

Using the properties of $z^{(1)}$ and $z^{(2)}$, and arguments similar to those used in the proof of theorem 4, we get

$$\frac{1}{2}\frac{d}{dt}|y|_H^2 + \nu\|y\|_V^2 \leq \frac{\nu}{2}\|y\|_V^2 + C_\nu \left|u^{(1)}\right|_H^2 |y|_H^2 + C_\nu\left(\left|u^{(1)}\right|_H + \left|u^{(2)}\right|_H\right)^2 \left|z^{(1)} - z^{(2)}\right|_H^2.$$

It is then easy to conclude by Gronwall lemma. ■

Consider now equation (3) and assume that $(W(t))_{t\geq 0}$ is a Brownian motion in H, with covariance Q, defined on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$. Since for P-a.e. $\omega \in \Omega$, the function $t \mapsto W_t(\omega)$ belongs to $C^\alpha([0,T];H)$ for every $\alpha \in \left(0, \frac{1}{2}\right)$, let us consider the following canonical situation: chosen $\alpha \in \left(0, \frac{1}{2}\right)$, take $\Omega = C^\alpha([0,T];H)$, with the Borel σ-field, the filtration associated to the canonical process $t \mapsto \omega(t)$, and the measure P given by the law of the previous Brownian motion $(W(t))_{t\geq 0}$. In this way the canonical process $t \mapsto \omega(t)$, is a Brownian motion in H, with covariance Q, having all paths of class $C^\alpha([0,T];H)$.

Let $u_0 : \Omega \to H$ be a given $\mathcal{F}_0$-measurable random variable. Given any path $t \mapsto \omega(t)$, of the Brownian motion described above, taken the corresponding initial condition $u_0(\omega)$, let $u(t,\omega)$ be the unique solution of (5), continuous in H. The function $(t,\omega) \mapsto u(t,\omega)$ is a stochastic process, $\mathcal{F}$-measurable for every $t \geq 0$, since $u(.,\omega)$ depends continuously on ω. It is also adapted since we may repeat the argument just given on any interval $[0,t]$, with arbitrary $t \geq 0$. Along with the pathwise uniqueness proved above, we have the first part of the following theorem:

Theorem 9 *Given an $\mathcal{F}_0$-measurable r.v. $u_0 : \Omega \to H$, there is a unique continuous in H adapted process $(u(t))_{t\geq 0}$ solution of equation (3). Moreover, if $E|u_0|_H^2 < \infty$ then*

$$E|u(t)|_H^2 + \int_0^t 2\nu E\|u(s)\|_V^2 ds = E|u_0|_H^2 + TrQ \cdot t \tag{12}$$

and

$$E\left[\sup_{t\in[0,T]}|u(t)|_H^2+\nu\int_0^T\|u(s)\|_V^2\,ds\right]\leq C\left(E|u_0|_H^2,TrQ,T\right). \tag{13}$$

If in addition $E|u_0|_H^p<\infty$ *for some* $p\geq 2$*, then*

$$E\left[\sup_{t\in[0,T]}|u(t)|_H^p\right]\leq C\left(p,E|u_0|_H^p,TrQ,T\right)$$

and

$$\frac{1}{T}E\int_0^T|u(s)|_H^p\,ds\leq C\left(p,TrQ,\nu,k_0\right)\left(1+\frac{E|u_0|_H^p}{T}\right). \tag{14}$$

Proof. The first claim has been proved above. We give only the proof of the part with $p\geq 2$ since the case $p=2$ is easier. To shorten a bit the notations we write u_t in place of $u(t)$ (we do not need the components here).

Let

$$\tau_R=\inf\left\{t\geq 0:|u_t|_H^2\geq R\right\}\wedge T,$$

and notice that $\tau_R\uparrow T$ as $R\to\infty$. We have

$$\begin{aligned}\pi_n u_{t\wedge\tau_R}&=\pi_n u_0+\int_0^{t\wedge\tau_R}\left[-\nu\pi_n Au_s-\pi_n B\left(u_s,u_s\right)\right]ds+\pi_n W_{t\wedge\tau_R}\\&=\pi_n u_0+\int_0^t\left[-\nu\pi_n Au_{s\wedge\tau_R}-\pi_n B\left(u_{s\wedge\tau_R},u_{s\wedge\tau_R}\right)\right]1_{s\leq\tau_R}ds\\&+\int_0^t 1_{s\leq\tau_R}\pi_n dW_s.\end{aligned}$$

From Itô formula, we have

$$\begin{aligned}&\sup_{t\in[0,\theta]}\left|\pi_n u_{t\wedge\tau_R}\right|^p+p\nu\int_0^\theta|\pi_n u_s|_H^{p-2}\left\langle\pi_n Au_s,\pi_n u_s\right\rangle_H 1_{s\leq\tau_R}ds\\&\quad\leq 2\left|\pi_n u_0\right|_H^p+2p\nu\int_0^\theta|\pi_n u_s|_H^{p-2}\left|\left\langle\pi_n B\left(u_s,u_s\right),\pi_n u_s\right\rangle_H\right|1_{s\leq\tau_R}ds\\&\quad+2\sup_{t\in[0,\theta]}\left|M_t^{(p)}\right|+p\left(p-1\right)Tr\left(\pi_n Q\right)\int_0^\theta|\pi_n u_s|_H^{p-2}1_{s\leq\tau_R}ds\end{aligned}$$

where
$$M_t^{(p)} = p \int_0^t 1_{s \le \tau_R} \left| \pi_n u_s \right|_H^{p-2} \langle \pi_n u_s, \pi_n dW_s \rangle_H .$$
By Burkholder-Davis-Gundy inequality
$$E \sup_{t\in[0,\theta]} \left| M_t^{(p)} \right|_H \le C(p, TrQ) E\left[\left(\int_0^\theta 1_{s\le\tau_R} \left|\pi_n u_s\right|_H^{2p-2} ds \right)^{1/2} \right]$$
$$\le \frac{1}{4} E\left[\sup_{t\in[0,\theta]} \left|\pi_n u_{t\wedge\tau_R}\right|_H^p \right] + C'(p, TrQ) E \int_0^\theta 1_{s\le\tau_R} \left|\pi_n u_s\right|_H^{p-2} ds$$
where all terms here and below are finite thanks to the stopping time. Hence
$$\frac{1}{2} E \sup_{t\in[0,\theta]} \left|\pi_n u_{t\wedge\tau_R}\right|_H^p + p\nu E \int_0^\theta \left|\pi_n u_s\right|_H^{p-2} \langle \pi_n A u_s, \pi_n u_s\rangle_H 1_{s\le\tau_R} ds$$
$$\le 2E\left|u_0\right|_H^p + 2p\nu E \int_0^\theta \left|\pi_n u_s\right|_H^{p-2} \left| \langle \pi_n B(u_s, u_s), \pi_n u_s \rangle_H \right| 1_{s\le\tau_R} ds$$
$$+ 2C'(p, TrQ) E \int_0^\theta 1_{s\le\tau_R} \left|\pi_n u_s\right|_H^{p-2} ds + p(p-1) TrQ \int_0^\theta \left|\pi_n u_s\right|_H^{p-2} 1_{s\le\tau_R} ds.$$
First we can take the limit as $n \to \infty$ (recall the uniform bound coming from the stopping time) and then deduce
$$\frac{1}{2} E \sup_{t\in[0,\theta]} \left|u_{t\wedge\tau_R}\right|_H^p \le E\left|u_0\right|_H^p + 1 + C''(p, TrQ) E \int_0^\theta 1_{s\le\tau_R} \sup_{t\in[0,s]} \left|u_{t\wedge\tau_R}\right|_H^p ds$$
which implies
$$E \sup_{t\in[0,T]} \left|u_{t\wedge\tau_R}\right|_H^p \le C\left(p, TrQ, E\left|u_0\right|_H^p\right)$$
by Gronwall lemma. By monotone convergence,
$$E \sup_{t\in[0,T]} \left|u_t\right|_H^p \le C\left(p, TrQ, E\left|u_0\right|_H^p\right).$$
Finally, from
$$k_0 p\nu E \int_0^\theta \left|u_s\right|_H^p 1_{s\le\tau_R} ds \le 2E\left|u_0\right|_H^p + 2C''(p, TrQ) E \int_0^\theta 1_{s\le\tau_R} \left|u_s\right|_H^{p-2} ds$$
we deduce
$$\frac{k_0 p\nu}{2} E \int_0^\theta \left|u_s\right|_H^p 1_{s\le\tau_R} ds \le 2E\left|u_0\right|_H^p + C(p, TrQ, \nu, k_0) \cdot \theta$$
and finally (14) with a new constant. The proof is complete. ■

2.4 Non-viscous limit

One of the peculiar features of the GOY model is that we may prove global existence of solutions for $\nu = 0$, even if in a sense this model resembles the 3D Euler equations, where such a global result is unknown. For the 2D Euler equations there is a second conserved quantity, the enstrophy, which yields a further a priori estimate on which the global existence of solutions is based. In 3D, such an estimate does not exists, and similarly it does not exist for the GOY model. But the nonlinear term of the GOY model is, in the analogy with particle systems, *finite-range*. Due to this fact, the unique a priori estimate of conservation of energy has stronger consequencies than in the case of Navier-Stokes equations and is sufficient to pass to the limit in the nonlinear term, and even for initial conditions of class H only (for 2D Euler equations one needs finite enstrophy initial conditions).

Let $\mathcal{D}$ be the set of all $\varphi \in H$ with compact support (namely with only a finite number of components being different from zero). Given

$$u_0 \in H, \quad \omega \in C^{\alpha}\left([0,T];H\right)$$

for some $\alpha > 0$, consider the *deterministic* equation with *zero viscosity*

$$\langle u(t), \varphi \rangle_H - \int_0^t \langle B\left(u(s), \varphi\right), u(s) \rangle_H \, ds = \langle u_0, \varphi \rangle_H + \langle \omega(t), \varphi \rangle_H \tag{15}$$

with $\varphi \in \mathcal{D}$. To prove the existence of solutions we have to assume some additional space regularity of the forcing term.

Theorem 10 *Given*

$$u_0 \in H, \quad \omega \in C^{\alpha}\left([0,T];H\right) \cap L^2(0,T;V)$$

for some $\alpha > 0$, there exists a solution u in $L^{\infty}(0,T;H)$ of equation (15). Moreover, for all its components u_n, we have $u_n \in C([0,T];\mathbb{R})$. Such a solution may be constructed as the weak star limit in $L^{\infty}(0,T;H)$ of a sequence $(u^{\nu_k})_{k\in\mathbb{N}}$ of solutions of equation (5) with viscosity $\nu_k > 0$, with the additional property that $u_n^{\nu_k}$ converges uniformly to u_n.

Proof. Step 1 (uniform estimates in $L^{\infty}(0,T;H)$). Let u^{ν} be the solution of equation (5) with viscosity $\nu > 0$. Let us set $v^{\nu} = u^{\nu} - \omega$ and omit the superscript ν in the intermediate computations. Then v is solution of the following problem

$$\frac{dv}{dt} + \nu A v = -B(v+\omega, v+\omega)dt - \nu A\omega, \tag{16}$$

Let us multiply this equation by v and, using the fact that $< B(v+\omega, v), v >= 0$, we get the following estimate (C denotes a generic constant that may take different values on different lines)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|v|_H^2 + \nu\|v\|_V^2 &\leq |< B(v+\omega,\omega), v >| + \nu\|\omega\|_V\|v\|_V \\ &\leq C|v|_H^2\|\omega\|_V + C|\omega|_H^2\|\omega\|_V + C\|\omega\|_V|v|_H^2 + \frac{\nu}{2}\|\omega\|_V^2 + \frac{\nu}{2}\|v\|_V^2.\end{aligned}$$

Using Gronwall Lemma, we get

$$\begin{aligned}|v(t)|_H^2 &\leq e^{\int_0^t (1+C\|\omega(s)\|_V)ds}|v_0|_H^2 \\ &\quad + \int_0^t e^{\int_s^t (1+C\|\omega(s)\|_V)dr}\left(C\,|\omega(s)|_H^2 + \nu\|\omega(s)\|_V\right)\|\omega(s)\|_V ds\end{aligned}$$

Here we can assume that there exists $\nu_0 > 0$ such that $\nu \leq \nu_0$. Hence, we get a uniform estimate on

$$|v^\nu(t)|_H^2 \leq C(\nu_0, T, |v_0|_H, \|\omega\|_{C([0,T];H)}, \|\omega\|_{L^2(0,T;V)}).$$

Hence, we get that also u^ν is uniformly bounded in $L^\infty(0,T;H)$.

Step 2 (uniform estimates on components in $C^\alpha([0,T];\mathbb{R})$). From the previous step, we know in particular that

$$\sup_{t\in[0,T]} |u_n^\nu(t)| \leq C$$

independently of n and $\nu \leq \nu_0$ (the independence on n is not even needed here). From equation (5) written componentwise

$$\begin{aligned}u_n^\nu(t) &= u_n^\nu(0) - \nu k_n^2 \int_0^t u_n^\nu(s)\,ds \\ &- \int_0^t ik_n\left(\frac{1}{4}\overline{u_{n-1}^\nu u_{n+1}^\nu} - \overline{u_{n+1}^\nu u_{n+2}^\nu} + \frac{1}{8}\overline{u_{n-1}^\nu u_{n-2}^\nu}\right)ds + \omega_n(t)\end{aligned} \tag{17}$$

we deduce that for every n there is a constant C_n, independent of $\nu \leq \nu_0$, such that

$$\|u_n\|_{C^\alpha([0,T];\mathbb{R})} \leq C_n.$$

By Ascoli-Arzelà theorem applied to every single n, there is a sequence $\left(\nu_k^{(n)}\right)_{k\in\mathbb{N}}$ going to zero such that $u_n^{\nu_k^{(n)}}$ converges uniformly to some $u_n^\infty \in C([0,T];\mathbb{R})$. By a diagonal procedure, we may chose a sequence $(\nu_k)_{k\in\mathbb{N}}$ independent of n such that $u_n^{\nu_k}$ converges uniformly to some $u_n \in C([0,T];\mathbb{R})$. From the bound of the previous step one has the weak star convergence in $L^\infty(0,T;H)$ of some subsequence of u^ν and one may take $(\nu_k)_{k\in\mathbb{N}}$ with both properties; the weak limit in $L^\infty(0,T;H)$ has clearly u_n as components, by a simple argument using the definition of weak star limit. Summarizing, we have a sequence $(\nu_k)_{k\in\mathbb{N}}$ and a function $u \in L^\infty(0,T;H)$ with components in $C([0,T];\mathbb{R})$ such that u^{ν_k} converges weak star to u in $L^\infty(0,T;H)$ and $u_n^{\nu_k}$ converges uniformly to u_n. From the uniform convergence it is easy to pass to the limit in the integral equation (17) and get (15). The proof is complete. ■

The uniqueness of solutions is an open problem. With a classical argument, see [1], one can prove the existence of a measurable selection in ω, thus the existence of a stochastic process that solves the stochastic zero-viscosity equation. However, the progressive measurability of a measurable selection is an open problem too.

3 Invariant measures

For every $x \in H$ there is a unique continuous adapted solution in H, call it $(u^x(t))_{t\geq 0}$. We also have (theorem 4) that, for every $t \geq 0$, if $x_n \to x$ in H then $u^{x_n}(t) \to u^x(t)$, P-a.s.

From these facts it is clear that the formula

$$(P_t\varphi)(x) = E\left[\varphi\left(u^x(t)\right)\right]$$

defines a mapping $P_t : B_b(H) \to B_b(H)$, which in addition has the Feller property $P_t(C_b(H)) \subset C_b(H)$ (by the dominated convergence theorem). One can show (as in [6]) that $(u^x(t))_{t\geq 0}$ defines a Markov process. We recall that a probability measure μ on H (endowed with the Borel σ-field) is invariant if

$$\mu(\varphi) = \mu(P_t\varphi)$$

for every $t \geq 0$ and $\varphi \in C_b(H)$.

Theorem 11 *There exists an invariant measure μ.*

Proof. Let us consider the solution with initial condition $x = 0$, that we denote by $(u^x(t))_{t\geq 0}$. Let ν_t be the law of $u^x(t)$ on H. Define, for every $T \geq 0$, the probability measure μ_T on H as

$$\mu_T = \frac{1}{T}\int_0^T \nu_s ds.$$

Since the semigroup P_t has the Feller property, if we prove that the family $\{\mu_T\}_{T\geq 0}$ is tight in H, then the existence of an invariant measure follows by the classical method of Krylov and Bogoliubov (see [6]). From (12) and the Chebishev inequality, we get

$$\begin{aligned}\mu_T\left(\|x\|_V^2 \geq R\right) &= \frac{1}{T}\int_0^T \nu_s\left(\|x\|_V^2 \geq R\right) ds \\ &\leq \frac{1}{T}\int_0^T \frac{E\left[\|u^x(s)\|_V^2\right]}{R} ds \leq \frac{C}{R}.\end{aligned}$$

This method is due to Chow and Hasminski [4]. The proof is complete. ■

The spatial regularity of invariant measures can be considerably improved; we do not develop this topic. On the other side, we shall use below the following fact:

Proposition 12 *For every $p \geq 2$ there is a constant $C(p, TrQ, \nu, k_0)$ such that every invariant measure μ satisfies*

$$\mu\left[\|\cdot\|_H^p\right] \leq C(p, TrQ, \nu, k_0).$$

Proof. Step 1. Let μ be an invariant measure. If $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$ is the filtered probability space where the Brownian motion is defined, consider the enlarged filtered probability space

$$\Omega' = \Omega \times H,\ \mathcal{F}' = \mathcal{F} \otimes \mathcal{B},\ \mathcal{F}'_t = \mathcal{F}_t \otimes \mathcal{B},\ P' = P \otimes \mu$$

with the new Brownian motion $(W'(t))$ and the $\mathcal{F}'_0$-measurable r.v. u_0 defined as

$$W'(t, \omega, x) = W(t, \omega),\ u_0(\omega, x) = x.$$

The law of u_0 is μ. The unique solution $(u(t))$ of (3) with initial condition u_0 is a stationary process, with the law of $u(t)$ equal to μ for every $t \geq 0$. We have to show that $E|u(t)|_H^p < \infty$.

Step 2. Given $\varepsilon > 0$, let $R_\varepsilon > 0$ be such that $P\left(|u_0|_H^p > R_\varepsilon\right) < \varepsilon$. Let $\Omega_\varepsilon \in \mathcal{F}$ be defined as $\Omega_\varepsilon = \{|u_0|_H^p \leq R_\varepsilon\}$; we have $P(\Omega_\varepsilon) \geq 1 - \varepsilon$. Define $u_0^{(\varepsilon)}$ as u_0 on Ω_ε, 0 otherwise. Let $\left(u^{(\varepsilon)}(t)\right)_{t\geq 0}$ be the unique solution of equation (3) with initial condition $u_0^{(\varepsilon)}$. Just looking at the integral form of (3) (which has an elementary pathwise meaning) it is easy to realize that $u^{(\varepsilon)}(\cdot,\omega) = u(\cdot,\omega)$ for P-a.e. $\omega \in \Omega_\varepsilon$. For $\left(u^{(\varepsilon)}(t)\right)_{t\geq 0}$ we have (14):

$$\begin{aligned}\frac{1}{T}E\int_0^T \left|u^{(\varepsilon)}(s)\right|_H^p ds &\leq C(p, TrQ, \nu, k_0)\left(1 + \frac{E\left[\left|u_0^{(\varepsilon)}\right|_H^p\right]}{T}\right)\\ &\leq C(p, TrQ, \nu, k_0)\left(1 + \frac{R_\varepsilon}{T}\right).\end{aligned}$$

Then, given $N > 0$,

$$\begin{aligned}&E\left(|u_0|_H^p \wedge N\right)\\ &= \frac{1}{T}\int_0^T E\left[|u(s)|_H^p \wedge N\right] ds\\ &= \frac{1}{T}\int_0^T E\left[1_{\Omega_\varepsilon}\left(|u(s)|_H^p \wedge N\right)\right] ds + \frac{1}{T}\int_0^T E\left[1_{\Omega_\varepsilon^c}\left(|u(s)|_H^p \wedge N\right)\right] ds\\ &\leq \frac{1}{T}\int_0^T E\left[1_{\Omega_\varepsilon}\left(\left|u^{(\varepsilon)}(s)\right|_H^p \wedge N\right)\right] ds + N\varepsilon\\ &\leq \frac{1}{T}\int_0^T E\left[\left|u^{(\varepsilon)}(s)\right|_H^p\right] ds + N\varepsilon\\ &\leq C(p, TrQ, \nu, k_0)\left(1 + \frac{R_\varepsilon}{T}\right) + N\varepsilon.\end{aligned}$$

It is now sufficient to take first the limit as $T \to \infty$, then as $\varepsilon \to 0$, finally as $N \to \infty$. The proof is complete. ■

Remark 13 *Similarly one can show that* $\mu\left[\|\cdot\|_V^2\right] < \infty$ *is true for every invariant measure.*

Finally, let us remark on the uniqueness and ergodicity of invariant measures. This is a rather technical topic, so we do not add it to this work. However, on the basis of the experience developed in the last ten years on this subject for stochastic Navier-Stokes equations, it is natural to expect that one can prove uniqueness and ergodicity, including exponential mixing, under the assumption that the noise forces a finite but sufficiently high number of modes. The proof can proceed by coupling of the low modes and Foias-Prodi inequalities to control the high modes. See [25] and references therein. In view of our application to K41 theory it would be interesting to prove a more difficult result, namely ergodicity when only very few modes are activated directly by the noise. There is hope to get such a result either for the finite dimensional Galerkin approximations, following [8], [26], or even for the infinite dimensional problem following [15]. However, especially the second result, that would fit with our framework, requires a long preparatory work of Malliavin calculus (see [24, 23]) that requires careful investigation beyond the scope of the present work.

3.1 Balance relations for invariant measures

Given $\nu > 0$, let μ^ν be any invariant measure of the GOY model and let E^ν be the corresponding expectation. We have proved that

$$E^\nu \left[|u|_H^p \right] \leq C \left(p, TrQ, \nu, k_0 \right)$$

hence in particular the mean quantities $E^\nu \left[|u_n|^2 \right]$ and $E^\nu \left[u_n u_{n+1} u_{n+2} \right]$ are well defined and finite. We have the following fundamental balance relation. This is one of the main advantages of the stochastic model: a rigorous and simple balance relation, with physical meaning, for expected values of basic quantities.

Proposition 14

$$\begin{aligned}
&\nu k_n^2 E^\nu \left[|u_n|^2 \right] + i2k_{n-1} \overline{E^\nu \left[u_n u_{n+1} u_{n+2} \right]} \\
&= ik_{n-3} \overline{E^\nu \left[u_{n-2} u_{n-1} u_n \right]} + ik_{n-2} \overline{E^\nu \left[u_{n-1} u_n u_{n+1} \right]} + \frac{Tr\left(\sigma_n \sigma_n^* \right)}{2}.
\end{aligned}$$

Proof. As in the proof of proposition 12, let $u(t)$ be a stationary solution associated to μ^ν. We may apply Itô formula componentwise and get

$$\begin{aligned}
\frac{1}{2} d |u_n|^2 &= \left(-\nu k_n^2 |u_n|^2 + ik_{n-3} \overline{u_{n-2} u_{n-1} u_n} + ik_{n-2} \overline{u_{n-1} u_n u_{n+1}} - 2ik_{n-1} \overline{u_n u_{n+1} u_{n+2}} \right) dt \\
&\quad + dM_n + \frac{Tr\left(\sigma_n \sigma_n^* \right)}{2} dt
\end{aligned}$$

where M_n is a square integrable martingale by the proved integrability properties of u and thus

$$E^\nu\left[|u_n(t)|^2\right]+\int_0^t \nu k_n^2 E^\nu\left[|u_n(s)|^2\right]ds+\int_0^t i2k_{n-1}\overline{E^\nu\left[u_n u_{n+1}u_{n+2}\right]}ds$$
$$=E^\nu\left[|u_n(0)|^2\right]+\int_0^t ik_{n-3}\overline{E^\nu\left[u_{n-2}u_{n-1}u_n\right]}ds$$
$$+\int_0^t ik_{n-2}\overline{E^\nu\left[u_{n-1}u_n u_{n+1}\right]}ds+\frac{Tr\left(\sigma_n\sigma_n^*\right)}{2}t.$$

By stationarity, $E^\nu\left[|u_n(t)|^2\right]=E^\nu\left[|u_n(0)|^2\right]$ and the integrands are independent of s. The balance relation readily follows. ■

The term $\nu k_n^2 E^\nu\left[|u_n|^2\right]$ has the meaning of mean rate of energy dissipation at scale k_n^{-1}; $i2k_{n-1}E^\nu\left[\overline{u_n u_{n+1}u_{n+2}}\right]$ may be interpreted as the mean rate of energy flux from scale k_n^{-1} to smaller scales, and similarly $ik_{n-3}E^\nu\left[\overline{u_{n-2}u_{n-1}u_n}\right]+$ $ik_{n-2}E^\nu\left[\overline{u_{n-1}u_n u_{n+1}}\right]$ as the mean rate of energy flux to scale k_n^{-1} from larger scales; $\frac{Tr(\sigma_n\sigma_n^*)}{2}$ is the mean rate of energy injection at scale k_n^{-1} due to external forces.

3.2 Symmetries of solutions and invariant measures

When the force acting on the GOY model is only white noise, as in this work, its symmetries reflect into some symmetries of solutions.

We know that the solution is pathwise unique (two solutions on the same stochastic basis coincide with probability one), so it is also unique in law (two solutions on two given stochastic basis have the same law). Therefore, given any sequence $S_n\in\{-1,1\}$, the unique solutions $u(t)$ and $v(t)$ of equations (2) and

$$\begin{aligned}&dv_n+\nu k_n^2 v_n dt\\&+ik_n\left(\frac{1}{4}\overline{v}_{n-1}\overline{v}_{n+1}-\overline{v}_{n+1}\overline{v}_{n+2}+\frac{1}{8}\overline{v}_{n-1}\overline{v}_{n-2}\right)dt\\&=S_n\sigma_n d\beta_n\end{aligned}\tag{18}$$

with equal-in-law initial conditions (in the sense that the initial conditions can be random but must have the same law) have the same law. Thus $E[u_n]=E[v_n]$ and so on. From this remark the proof of the following statement is obvious.

Any sequence $\{S_n\}$ in $\{-1,1\}$ induces a mapping $S : H \to H$ defined as $(Su)_n = S_n u_n$.

Proposition 15 *Let $\{S_n\}$ be a sequence in $\{-1,1\}$ of period 3 ($S_{n+3} = S_n$) with $S_n S_{n+1} S_{n+2} = 1$ and let S be the associated mapping. Let $u(t)$ be the solution with initial condition u^0 and let $v(t)$ be defined as $v(t) = Su(t)$. Then $v(t)$ is the solution of (18) with initial condition $v^0 = Su^0$.*

Lemma 16 *If in addition u^0 and v^0 have the same law, then also u and v have the same law.*

Corollary 17 *Let $u(t)$ be the solution with initial condition 0. Then*

$$E[u_n(t)] = 0, \quad E[u_n(t)\, u_{n+1}(t)] = 0 \text{ for every } n.$$

This result can be generalized to other initial conditions and other moments (not all!), but we limit ourselves to the previous cases as an illustration. Concerning invariant measures we have:

Proposition 18 *There exists an invariant measure μ of the GOY equations with the properties*

$$E^{\mu}[u_n] = 0, \quad E^{\mu}[u_n u_{n+1}] = 0 \text{ for every } n.$$

Proof. With the notations of the proof of theorem 11, Let ν_t be the law of $u^0(t)$ (zero initial condition) and let $\mu_T = \frac{1}{T}\int_0^T \nu_s ds$. The expected value of u_n and $u_n u_{n+1}$ under μ_T is zero, by the previous corollary. This property is stable under weak limit, so it is satisfied by the invariant measure constructed in the proof of theorem 11. The proof is complete. ■

Of course if we have uniqueness of invariant measures, the unique invariant measure has the property stated in the proposition. However we cannot prove that such property is true for every invariant measures in case of non-uniqueness: we cannot exclude a symmetry breaking.

4 Remarks on K41 theory

4.1 Definitions

For every $\nu > 0$, let μ^{ν} be any invariant measure of the GOY model as above. We have proved above that $E^{\nu}[|u|_H^p] < \infty$ for every $p \geq 0$.

The expression

$$S_p^\nu(n) := E\left[|u_n^\nu|^p\right]$$

which is finite, is called the *p-order structure function*. It is considered, for the GOY model, as the analog of $S_p^\nu(r) = E\left[\|v(r\cdot e) - v(0)\|^p\right]$, where $v(x)$ is a stationary isotropic random field describing a 3D turbulent fluid; the correspondence is through $r = k_n^{-1}$.

One is interested in a scaling behavior of the form

$$E\left[|u_n^\nu|^p\right] \sim k_n^{-\zeta_p}.$$

This can be true only in an *intermediate* range of n's, since for n large enough the decay is the one of regular functions. So we look for a range of the form $n \in [n_-(\nu), n_+(\nu)]$, with $n_-, n_+ : (0,1) \to \mathbb{N}$ such that $n_-(\nu) < n_+(\nu)$, $\lim_{\nu\to 0} \frac{n_-(\nu)}{n_+(\nu)} = 0$, and sometimes $\lim_{\nu\to 0} n_-(\nu) = \infty$. There is no unique prescription for n_-, n_+, but following Kolmogorov and dimensional analysis one has the idea that $n_+(\nu)$ should roughly satisfy

$$\lim_{\nu\to 0} \frac{n_+(\nu)}{\log_2 \nu} = -\frac{3}{4} \tag{19}$$

corresponding to the idea $k_{n_+(\nu)}^{-1} \sim \nu^{3/4}$. In fact we shall work on a slightly reduced range, like $k_{n_+(\nu)}^{-1} \sim \nu^{3/4-\varepsilon}$ for some $\varepsilon > 0$.

Definition 19 *Let $(\mu^\nu)_{\nu>0}$ be any family of invariant measures of the GOY model. Let $n_-, n_+ : (0,1) \to \mathbb{N}$ such that $n_-(\nu) < n_+(\nu)$ and $\lim_{\nu\to 0} \frac{n_-(\nu)}{n_+(\nu)} = 0$ and let*

$$R = \{(\nu, n) \in (0,1) \times \mathbb{N} : n \in [n_-(\nu), n_+(\nu)]\}.$$

Given $p \geq 0$, we call sub and super asymptotic exponents of order p, for the family $(\mu^\nu)_{\nu>0}$ relative to the range $[n_-(\nu), n_+(\nu)]$, the numbers (possibly infinite)

$$\zeta_p^+ := -\liminf_{\substack{\nu\to 0\\ (\nu,n)\in R}} \frac{1}{n} \log_2 E^\nu\left[|u_n|^p\right]$$

$$\zeta_p^- := -\limsup_{\substack{\nu\to 0\\ (\nu,n)\in R}} \frac{1}{n} \log_2 E^\nu\left[|u_n|^p\right].$$

Clearly $\zeta_p^- \leq \zeta_p^+$. If $\zeta_p^- = \zeta_p^+$ we call the common value ζ_p the asymptotic exponent of order p.

We could say that $(\mu^\nu)_{\nu>0}$ has the K41 scaling property if

$$\zeta_2 = \frac{2}{3} \text{ with } n_+ \text{ satisfying (19).}$$

It is useful to introduce another notation.

Definition 20 *Standing the previous notations, we call flux asymptotic exponent the number*

$$\zeta_3^{flux} := -\lim_{\substack{\nu\to 0\\ (\nu,n)\in R}} \frac{1}{n}\log_2\left|E^\nu\left[u_n u_{n+1} u_{n+2}\right]\right|$$

when it exists.

The number ζ_3^{flux} clearly resembles ζ_3, at the dimensional level at least, but we cannot prove they are equal. The superscript "flux" should not be misunderstood: the mean rate of flux is described by $ik_n E^\nu\left[\overline{u_n u_{n+1} u_{n+2}}\right]$, thus its asymptotic exponent is $\zeta_3 - 1$.

The function $p \mapsto \zeta_p$ looks like a free energy associated to a one-dimensional particle system with energy $-\log|u_n|$, but this analogy may be dangerous since we are not dealing with an equilibrium system; let us mention it only at formal level as a technical tool. In spite of this similarity, it is not easy to prove the existence of ζ_p by classical arguments like sub-additivity. Large deviations of $\frac{\log|u_n|}{n}$ and Gartner-Ellis theorem are also heuristically involved, see also the number α mentioned in (20), but its rigorous use is another open problem.

4.2 General facts on asymptotic exponents

The elementary (rigorous) results of this section do not depend on the GOY equation but just on the definition of asymptotic exponents. They give us a reference picture over which we can put the particular results due to the balance law of the GOY model.

In section 5 we shall prove results about the claim $\zeta_3^{flux} = 1$. So it is interesting to know at least one (obvious) inequality between ζ_3^{flux} and ζ_3^- or ζ_3.

Lemma 21 *If ζ_3^{flux} exists, then*

$$\zeta_3^- \leq \zeta_3^{flux}.$$

If in addition also ζ_3 exist, then

$$\zeta_3 \leq \zeta_3^{flux}.$$

Proof. By Hölder inequality

$$|E^\nu [u_n u_{n+1} u_{n+2}]| \leq \prod_{j=0}^{2} E^\nu \left[|u_{n+j}|^3\right]^{1/3}$$

hence

$$\frac{1}{n} \log_2 |E^\nu [u_n u_{n+1} u_{n+2}]| \leq \frac{1}{3} \sum_{j=0}^{2} \frac{1}{n} \log_2 E^\nu \left[|u_{n+j}|^3\right].$$

The result easily follows. ■

Lemma 22 *Given a range $[n_-(\nu), n_+(\nu)]$, the function $p \mapsto \zeta_p^-$ is concave over $[0,\infty)$.*

If the function $p \mapsto \zeta_p$ is well defined on some interval, relative to the same range $[n_-(\nu), n_+(\nu)]$, then it is concave on that interval.

Proof. Again by Hölder inequality we have

$$E^\nu \left[|u_n|^{\alpha x+(1-\alpha)y}\right] \leq E^\nu [|u_n|^x]^\alpha E^\nu [|u_n|^y]^{1-\alpha}$$

for every $x, y \geq 0$ and $\alpha \in (0,1)$. Hence

$$\frac{1}{n} \log_2 E^\nu \left[|u_n|^{\alpha x+(1-\alpha)y}\right] \leq \alpha \frac{1}{n} \log_2 E^\nu [|u_n|^x] + (1-\alpha) \frac{1}{n} \log_2 E^\nu [|u_n|^y].$$

It is now sufficient to take the limit as $\nu \to 0$ constrained on R. ■

We do not know yet how to get informations on ζ_3^- from the GOY model, otherwise, thanks to the previous concavity result, this would be another route to have criteria for the result $\zeta_2^- \geq \frac{2}{3}$ (investigated below in section 5):

$$\zeta_2^- \geq \frac{2}{3}\zeta_3^-$$

(since $\zeta_0^- = 0$).

One can write several relations between the asymptotic exponents and ratios of moments. The following one will be used below.

Lemma 23 *Let the range* $[n_-(\nu), n_+(\nu)]$ *be given, with*

$$\nu^{-\varepsilon} \leq k_{n_-(\nu)} < k_{n_+(\nu)} \leq \nu^{-\alpha}$$

for some $\alpha > \varepsilon > 0$ *and sufficiently small* ν. *Assume*

$$\zeta_3^{flux} = 1.$$

Then

$$\zeta_2^- \geq \frac{2}{3} \quad \Leftrightarrow \quad \limsup_{\substack{\nu\to 0\\(\nu,n)\in R}} \frac{1}{\log \nu^{-1}} \log \frac{E^\nu\left[|u_n|^2\right]}{|E^\nu\left[u_n u_{n+1} u_{n+2}\right]|^{2/3}} \leq 0$$

and

$$\zeta_2 = \frac{2}{3} \quad \Leftrightarrow \quad \lim_{\substack{\nu\to 0\\(\nu,n)\in R}} \frac{1}{\log \nu^{-1}} \log \frac{E^\nu\left[|u_n|^2\right]}{|E^\nu\left[u_n u_{n+1} u_{n+2}\right]|^{2/3}} = 0.$$

Remark 24 *Although obvious, let us make explicit that the right-hand-side condition in the first equivalence means that for every* $\varepsilon > 0$ *there is* $\nu_0 > 0$ *such that*

$$\frac{E^\nu\left[|u_n|^2\right]}{|E^\nu\left[u_n u_{n+1} u_{n+2}\right]|^{2/3}} \leq \nu^{-\varepsilon}$$

for every $(\nu, n) \in (0, \nu_0] \times [n_-(\nu), n_+(\nu)]$. *For instance, this is fulfilled if*

$$\frac{E^\nu\left[|u_n|^2\right]}{|E^\nu\left[u_n u_{n+1} u_{n+2}\right]|^{2/3}} \leq C \log \nu^{-1}$$

for $(\nu, n) \in (0, \overline{\nu}] \times [n_-(\nu), n_+(\nu)]$ *with some* $\overline{\nu} > 0$, $C > 0$.

Remark 25 *Similarly, the right-hand-side condition in the second equivalence means that for every* $\varepsilon > 0$ *there is* $\nu_0 > 0$ *such that*

$$\nu^{\varepsilon} \leq \frac{E^\nu\left[|u_n|^2\right]}{|E^\nu\left[u_n u_{n+1} u_{n+2}\right]|^{2/3}} \leq \nu^{-\varepsilon}$$

for every $(\nu, n) \in (0, \nu_0] \times [n_-(\nu), n_+(\nu)]$. *For instance, this is fulfilled if*

$$\frac{C_1}{\log \nu^{-1}} \leq \frac{E^\nu\left[|u_n|^2\right]}{|E^\nu\left[u_n u_{n+1} u_{n+2}\right]|^{2/3}} \leq C_2 \log \nu^{-1}$$

for $(\nu, n) \in (0, \overline{\nu}] \times [n_-(\nu), n_+(\nu)]$ *with some* $\overline{\nu} > 0$, $C_1, C_2 > 0$.

Proof. Let us prove the implication $\Rightarrow$ of the first equivalence. Given $\varepsilon > 0$, from $\zeta_2^- \geq \frac{2}{3}$ and $\zeta_3^{flux} = 1$ there is $\nu_0 > 0$ such that

$$E^\nu\left[|u_n|^2\right] \leq k_n^{-\frac{2}{3}+\varepsilon}$$
$$k_n^{-1-\varepsilon} \leq |E^\nu\left[u_n u_{n+1} u_{n+2}\right]|$$

for $(\nu, n) \in (0, \nu_0] \times [n_-(\nu), n_+(\nu)]$, thus

$$\frac{E^\nu\left[|u_n|^2\right]}{|E^\nu\left[u_n u_{n+1} u_{n+2}\right]|^{2/3}} \leq k_n^{5\varepsilon/3} \leq k_{n_+(\nu)}^{5\varepsilon/3} \leq \nu^{-\alpha\varepsilon}.$$

The result easily follows.

Conversely, given $\varepsilon > 0$, there is $\nu_0 > 0$ such that

$$E^\nu\left[|u_n|^2\right] \leq \nu^{-\varepsilon} |E^\nu\left[u_n u_{n+1} u_{n+2}\right]|^{2/3}$$
$$|E^\nu\left[u_n u_{n+1} u_{n+2}\right]| \leq k_n^{-1+\varepsilon}$$

for $(\nu, n) \in (0, \nu_0] \times [n_-(\nu), n_+(\nu)]$, thus

$$E^\nu\left[|u_n|^2\right] \leq \nu^{-2\varepsilon/3} k_n^{(-1+\varepsilon)2/3}$$

$$\limsup_{\substack{\nu\to 0\\(\nu,n)\in R}} \frac{1}{n} \log_2 E^\nu\left[|u_n|^2\right] \leq (-1+\varepsilon)\frac{2}{3} + \varepsilon \limsup_{\nu\to 0} \frac{1}{n_-(\nu)} \log_2 \nu^{-1}.$$

The result follows from the assumption on $n_-(\nu)$. The proof of the first equivalence is complete and the proof of the second one is analogous. ■

Remark 26 *One of the main conceptual results of the present work is that, due to the GOY equation balance laws, the equivalences stated by the previous lemma are true (up to some detail on* $[n_-(\nu), n_+(\nu)]$*) without the assumption* $\zeta_3^{flux} = 1$*. Indeed, both conditions on the two sides of both the equivalences imply* $\zeta_3^{flux} = 1$ *(when* $n_-(\nu) = 1$*, using the GOY balance laws).*

4.3 Comments on the value of ζ_2

Kolmogorov [17] conjectured the value $\zeta_2 = \frac{2}{3}$ for isotropic fully turbulent 3D fluids. The assumptions at the basis of his work would also give $\zeta_p = \frac{p}{3}$. All numerical experiments on the GOY model (as well as experiments on real

3D fluids) indicate a strong deviation from this value for large p, and in the direction

$$\zeta_p < \frac{p}{3} \text{ for } p > 3.$$

On the value

$$\zeta_3 = 1$$

there is also a general agreement. *Some* numerical experiment on the GOY model give also

$$\zeta_2 > \frac{2}{3}$$

(ζ_2 close to 0.7), but there is also some controversy on this deviation. Finally, notice that $\zeta_0 = 0$. A natural question is thus whether

- ζ_p is a strictly concave function over the whole range $p \geq 0$, and $\zeta_2 > \frac{2}{3}$ or
- ζ_p is a line on $p \in [0, p^*]$, $p^* \geq 3$ (with presumably $p^* = 3$), so that $\zeta_2 = \frac{2}{3}$, and ζ_p is strictly concave for $p > p^*$

(there are also other possibilities of course, we have mentioned only the most natural two of them). If it exists, the derivative at zero,

$$\alpha := \frac{d\zeta_p}{dp}|_{p=0} \tag{20}$$

is also an interesting quantity, related to large deviation properties of $\frac{\log|u_n|}{n}$. Part of the previous question is whether

$$\alpha = \frac{2}{3} \quad \text{or} \quad \alpha > \frac{2}{3}.$$

The answer to these questions contains striking informations on the dynamics and is not just the minor difference between 0.66 and 0.7: see section 4.4 below.

The aim of this paper is modest with respect to these difficult questions. First, we give rigorous criterions to establish that

$$\zeta_3^{flux} = 1$$

which is in a sense similar to the (unproved) belief $\zeta_3 = 1$. The criterion is based on an assumption on the behavior in ν of the quantity

$$\sup_{n \in [n_-(\nu), n_+(\nu)]} \frac{E^\nu \left[|u_n|^2 \right]}{\left| E^\nu \left[u_n u_{n+1} u_{n+2} \right] \right|^{2/3}}. \tag{21}$$

We cannot prove that this assumption is satisfied but the numerical investigation of its validity looks more robust than the fit of scaling exponents. In addition, the reformulation of scaling problems in terms of such kind of assumptions looks appealing for future investigations (it could be related to the structure of statistical dependence of the variables u_n, u_{n+1}, u_{n+2} on which we hope one can throw some light).

Second, as a consequence of the previous fact (always under the previous assumption on (21)), we get $\zeta_2^- \geq \frac{2}{3}$ and thus $\zeta_2 \geq \frac{2}{3}$ if ζ_2 exists. This is not a new information but our aim is to start to construct rigorous proofs of segments of the theory.

Finally, we also have a criterion, again based on $\frac{E^\nu\left[|u_n|^2\right]}{\left|E^\nu[u_n u_{n+1} u_{n+2}]\right|^{2/3}}$, to investigate the question $\zeta_2 = \frac{2}{3}$. However, it is not clear whether our numerical results support this result.

4.4 Dynamical relevance of $\zeta_2 > \frac{2}{3}$ against $\zeta_2 = \frac{2}{3}$

The difference between the two cases $\zeta_2 > \frac{2}{3}$ and $\zeta_2 = \frac{2}{3}$, although maybe quantitatively small, is the signature of a great difference in dynamical behavior.

The result $\zeta_2 = 2/3$ is the most innocent one from the dynamical viewpoint. The configuration $u_n = C \cdot k_n^{-1/3}$ is approximatively a solution, unstable, and the system fluctuates not far from it most of the time, except for bursts that produce anomalies in moments of high order but not for $p = 2$ (and 3). What is not clear in this picture is the discrepancy with the simulations/fits that gave results like $\zeta_2 = 0.7$, and the presumable phase transition associated to the change in shape of the function $p \mapsto \zeta_p$ (from the straight line $p \mapsto p/3$ at least for $0 \leq p \leq 3$ to a strictly concave function for large p), remarkable for a one dimensional system.

The result $\zeta_2 > 2/3$ is, on the other side, quite intriguing from the dynamical point of view. Most of the time $|u_n|$ should be exponentially smaller than $k_n^{-1/3}$. Since we believe that $\zeta_3 = 1$, the burst should be such that their value and probability compensate exactly to produce the integer exponent of order three. However the latter miracle could just be a consequence of a flow balance equations. But what could produce values of $|u_n|$ exponentially smaller than $k_n^{-1/3}$ with overwhelming probability? One possibility is a locking phenomenon easily observed in the numerical evolution of the configuration $(u_n)_{n\geq 1}$. Most of the configurations have a general tendency to shift to the right, with a progressive direct cascade of energy. But certain configurations with a change of sign (properly interpreted in the complex plane) have a tendency to rest and act as a sink of energy from both sides (namely with a direct and inverse local cascade around them), so that the other components have typically a smaller value of energy than the one of a normal continuous flow. Suddenly, when a certain threshold is reached, these locking configurations start to move and carry energy to high frequencies. Dissipation, which is quite low during the locking phase, has a burst when the wave of energy reaches the dissipation range. This is in agreement with the bursty records of dissipation.

The picture in the case $\zeta_2^- > 2/3$ is by no means that of a continuous direct cascade (up to fluctuations), but that of emergence of structures that absorb energy from nearby modes (in a direct and inverse fashion), keep quite stably the energy at a given mode, and then suddenly have a spike. The analogy with the idea of coherent vortex structures is strong, even if it is questionable whether the three dimensionality of the space may offer the necessary degrees of freedom to unlock the resting configurations much faster than in the GOY model. Thus we believe that even if the truth is $\zeta_2^- > 2/3$ for the GOY model, this is not one of the result that presumably should be easily transferred to real fluids.

There is however at least a third possibility. In our numerical simulations we observe a modest extra value of steepness with respect to the slope 2/3, but it decreases with ν and becomes unclear due to computational limitations. So the correct limit result could be $\zeta_2 = 2/3$ but numerics for finite ν undoubtedly show a correction. On one side, this correction should be captured by a proper definition, which is not our definition of ζ_2. On the other side, one should understand the dynamical reason for this correction, that could be related to the explanation given above of the case $\zeta_2 > 2/3$.

5 Rigorous results

To avoid general conditions, we impose throughout this section the following assumption on the noise:

$$\sigma_1 = ... = \sigma_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}, \sigma_n = 0 \text{ for } n > 4. \tag{22}$$

With minor quantitative change the results are the same with a finite number of σ_n different from zero, with at least nonzero σ_1 and σ_2.

On the range $[n_-(\nu), n_+(\nu)]$ we need to impose that

$$k_{n_+(\nu)} << \nu^{-3/4}.$$

We need a separation, although arbitrarily small, between the order of $k_{n_+(\nu)}$ in ν and the order 3/4. One way to express this condition is

$$\liminf_{\nu\to 0} \frac{n_+(\nu)}{\log_2 \nu} > -\frac{3}{4}. \tag{23}$$

Let us stress again that (22) and (23) will be standing assumptions in this subsection. This will not be repeated in the main statements.

Concerning $n_-(\nu)$, sometimes it is essential to assume $n_-(\nu) = 1$ in order to start an iteration on the balance laws (14). In other cases we need a slightly diverging $n_-(\nu)$ to get rid of minor divergences. The usual assumption in such a case will be

$$\limsup_{\nu\to 0} \frac{\log_2 \nu^{-1}}{n_-(\nu)} < \infty. \tag{24}$$

Let $(\mu^\nu)_{\nu>0}$ be a family of stationary measures of the GOY model depending on the viscosity and let E^ν denote the corresponding expectation. Notice that all the next results do not depend on the measures we choose, so possible non-uniqueness of invariant measures does not affect our results.

Remark 27 *In general, $E^\nu[u_n u_{n+1} u_{n+2}]$ are pure imaginary numbers. All terms in the energy balance at mode n are real. Moreover,*

$$i \cdot \overline{E^\nu[u_n u_{n+1} u_{n+2}]} = |E^\nu[u_n u_{n+1} u_{n+2}]| \, sgn(-iE^\nu[u_n u_{n+1} u_{n+2}]).$$

The proof of this fact is easy from the balance relations (14) and an iterative procedure.

Lemma 28 *The inequality*

$$i\cdot\overline{E^{\nu}\left[u_n u_{n+1} u_{n+2}\right]}\leq 2\sigma^2 k_n^{-1}$$

is true for every n*.*

Proof. From the balance relations (14), with the notations

$$\phi_n = ik_{n-1}\overline{E^{\nu}\left[u_n u_{n+1} u_{n+2}\right]},\quad \epsilon_n=\nu k_n^2 E^{\nu}\left[\left|u_n\right|^2\right]$$

we have

$$\phi_n+\epsilon_n=\frac{\phi_{n-2}+\phi_{n-1}}{2}+\frac{Tr\left(\sigma_n\sigma_n^*\right)}{4}. \tag{25}$$

Notice that all these numbers are real.

From (25) and $\epsilon_n \geq 0$ we readily have

$$\phi_n\leq\frac{\phi_{n-2}+\phi_{n-1}}{2}+\frac{\sigma_n^2}{4}.$$

Taking into account that $\phi_{-1}=\phi_0=0$ and assumption (22), we easily bound the first 4 terms ϕ_1, ... , ϕ_4 by

$$\frac{\sigma^2}{4},\quad\frac{3\sigma^2}{8},\quad\frac{9\sigma^2}{16},\quad\frac{23\sigma^2}{32}$$

and thus, simplifying,

$$\phi_n\leq\sigma^2 \text{ for every } n.$$

The proof is complete. ■

We start with the simplest form of our main result. It is not entirely a particular case of the next one since it also give us stronger results. We are not sure that its assumption is satisfied by our numerics (contrary to the assumption of the next theorem), so we give it mainly for pedagogical reasons, since its conditions are easier to read.

Theorem 29 *Assume that for some* $\overline{\nu}>0$ *and* $\gamma>0$ *we have*

$$\frac{E^{\nu}\left[\left|u_n\right|^2\right]}{\left|E^{\nu}\left[u_n u_{n+1} u_{n+2}\right]\right|^{2/3}}\leq\gamma$$

for every $(\nu,n)\in(0,\overline{\nu}]\times[1,n_+(\nu)]$*. Then there are constants* $C>0$ *and* $\nu_0>0$*, depending only on* σ*,* γ *and* n_+ *such that*

$$Ck_n^{-1}\leq i\cdot\overline{E^{\nu}\left[u_n u_{n+1} u_{n+2}\right]}\leq 2\sigma^2 k_n^{-1}$$

and

$$Ck_n^{-1} \leq |E^\nu [u_n u_{n+1} u_{n+2}]| \leq 2\sigma^2 k_n^{-1} \tag{26}$$

for $(\nu, n) \in (0, \nu_0] \times [1, n_+(\nu)]$ *(the right-hand-side inequality is true for every* n*). Hence also, a fortiori,*

$$E^\nu \left[|u_n|^2\right] \leq 2\gamma\sigma^2 k_n^{-2/3}$$

for $(\nu, n) \in (0, \nu_0] \times [1, n_+(\nu)]$*. Therefore, with respect to any range of the form* $[n_-(\nu), n_+(\nu)]$ *with diverging* $n_-(\nu)$*, we have*

$$\zeta_2^- \geq \frac{2}{3}.$$

In addition, the mean dissipation rate is exponentially smaller than the mean flux rate for $(\nu, n) \in (0, \nu_0] \times [1, n_+(\nu)]$*, as stated in claim 30.*

Proof. Step 1 (estimates on ϵ_n). We use the notations of the previous lemma. We prove the following claim:

Claim 30 *there is* $\alpha \in (0, 1)$ *depending only on* n_+ *and given any* $\delta > 0$ *there is* $\nu_0 > 0$ *depending only on* δ*,* γ *and* n_+*, such that*

$$\epsilon_n \leq \delta\alpha^n |\phi_n|^{2/3}$$

for $(\nu, n) \in (0, \nu_0] \times [1, n_+(\nu)]$*.*

Under our assumptions, for $1 \leq n \leq n_+(\nu)$ we have

$$\begin{aligned} \epsilon_n &\leq \nu k_n^2 \gamma |E[u_n u_{n+1} u_{n+2}]|^{2/3} = 2^{-4/3}\gamma\nu k_n^{4/3} |\phi_n|^{2/3} \\ &\leq \left(2^{-4/3}\gamma\nu^{\varepsilon_1}\right)\left(\nu^{1-\varepsilon_1} k_{n_+(\nu)}^{4/3+\varepsilon_2}\right) k_n^{-\varepsilon_2} |\phi_n|^{2/3} \end{aligned}$$

for every $\varepsilon_1, \varepsilon_2 > 0$. From the assumption on $n_+(\nu)$, there exist $\varepsilon > 0$ and $\nu_0' > 0$ such that

$$\frac{n_+(\nu)}{\log_2 \nu} \geq -\frac{3}{4} + \varepsilon$$

for every $\nu \in (0, \nu_0')$. Hence $\nu^{\frac{3}{4}-\varepsilon} 2^{n_+(\nu)} \leq 1$ for $\nu \in (0, \nu_0')$, and thus, if we choose $\varepsilon_1, \varepsilon_2 > 0$ such that

$$\frac{1 - \varepsilon_1}{4/3 + \varepsilon_2} = \frac{3}{4} - \varepsilon$$

we have

$$\nu^{1-\varepsilon_1} k_{n_+(\nu)}^{4/3+\varepsilon_2} = \left(\nu^{\frac{1-\varepsilon_1}{4/3+\varepsilon_2}} 2^{n_+(\nu)}\right)^{4/3+\varepsilon_2} \leq 1.$$

We have proved that for $\nu \in (0, \nu_0')$ and $1 \leq n \leq n_+(\nu)$

$$\epsilon_n \leq \left(2^{-4/3}\gamma\nu^{\varepsilon_1}\right) k_n^{-\varepsilon_2} |\phi_n|^{2/3}.$$

The claim 30 is an easy consequence of this result, with new symbols.

Step 2 (estimate from below for $\phi_1, ..., \phi_4$). For future reference, let us stress that all steps from now on will depend only on (25) and claim 30, so they remain true under more general assumptions when these two ingredients are true.

Choose $\delta \leq \frac{\sigma^2}{40}$ (for instance). Let us work for $(\nu, n) \in (0, \nu_0] \times [1, n_+(\nu)]$.

From (25) we have

$$\phi_1 + \epsilon_1 = \frac{\sigma^2}{4}$$

where $\epsilon_1 \leq \delta\alpha |\phi_1|^{2/3}$ (claim 30). Hence

$$\phi_1 + \delta\alpha |\phi_1|^{2/3} \geq \frac{\sigma^2}{4}.$$

By the smallness of δ, the maximum of the function $x \mapsto x + \delta\alpha |x|^{2/3}$ for $x < 0$ is smaller than $\frac{\sigma^2}{4}$. Hence we get

$$\phi_1 \geq 0.$$

Together with the bound of step 3, this implies $|\phi_1| \leq \sigma^2$ and thus $\epsilon_1 \leq \delta\alpha\sigma^{4/3}$. Therefore

$$\phi_1 \geq \frac{\sigma^2}{4} - \delta\alpha\sigma^{4/3}.$$

For $n = 2$, from (25) we have

$$\phi_2 + \epsilon_2 = \frac{\phi_1}{2} + \frac{\sigma^2}{4}$$

where $\epsilon_2 \leq \delta\alpha^2 |\phi_2|^{2/3}$ (claim 30), so

$$\phi_2 + \delta\alpha^2 |\phi_2|^{2/3} \geq \frac{\phi_1}{2} + \frac{\sigma^2}{4} \geq \frac{\sigma^2}{4}.$$

As above we get

$$\phi_2 \geq \frac{\sigma^2}{4} - \delta\alpha^2\sigma^{4/3}.$$

The same is true for ϕ_3 and ϕ_4, with α^3 and α^4, so we have proved:

$$\phi_n \geq \frac{\sigma^2}{4} - \delta\alpha^n\sigma^{4/3} \text{ for } n = 1, 2, 3, 4.$$

One could prove a much better bound, but qualitatively this is sufficient.

Step 3 (estimate from below for ϕ_n, $n \in [5, n_+(\nu)]$). In addition to $\delta \leq \frac{\sigma^2}{40}$ we also require that

$$\frac{\delta\sigma^{4/3}}{1-\alpha} \leq \frac{\sigma^2}{8}.$$

We prove by induction that

$$\phi_n \geq \frac{\sigma^2}{4} - \delta\sigma^{4/3}\left(1 + \alpha + ... + \alpha^n\right)$$

for $(\nu, n) \in (0, \nu_0] \times [1, n_+(\nu)]$. This will complete the proof. This inequality is true for $n = 1, ..., 4$. Let us assume it is true for $n = 1, ..., k$, for some $k \geq 4$, and let us prove it for $n = k + 1$. From (25) for $n = k + 1$ we have

$$\phi_{k+1} + \epsilon_{k+1} = \frac{\phi_{k-1} + \phi_k}{2}$$

with $\epsilon_{k+1} \leq \delta\alpha^{k+1}\left|\phi_{k+1}\right|^{2/3}$ (claim 30), so

$$\phi_{k+1} + \delta\alpha^{k+1}\left|\phi_{k+1}\right|^{2/3} \geq \frac{\phi_{k-1} + \phi_k}{2}$$

$$\begin{aligned} &\geq \frac{\sigma^2}{4} - \delta\sigma^{4/3}\frac{\left(1 + \alpha + ... + \alpha^{k-1}\right) + \left(1 + \alpha + ... + \alpha^k\right)}{2} \\ &\geq \frac{\sigma^2}{4} - \delta\sigma^{4/3}\left(1 + \alpha + ... + \alpha^k\right) \geq \frac{\sigma^2}{8}. \end{aligned}$$

Again by the smallness of δ we have $\phi_{k+1} \geq 0$, hence $\left|\phi_{k+1}\right| \leq \sigma^2$, $\epsilon_{k+1} \leq \delta\alpha^{k+1}\sigma^{4/3}$,

$$\begin{aligned} \phi_{k+1} &\geq \frac{\sigma^2}{4} - \delta\sigma^{4/3}\left(1 + \alpha + ... + \alpha^k\right) - \delta\alpha^{k+1}\sigma^{4/3} \\ &= \frac{\sigma^2}{4} - \delta\sigma^{4/3}\left(1 + \alpha + ... + \alpha^{k+1}\right). \end{aligned}$$

The proof is complete. ■

Since the results of the previous theorem depend only on (25) and claim 30, the assumptions can be generalized, at the price of a less intuitive statement. The simplest generalization would be to assume that there is a function $\gamma : (0,1) \rightarrow (0,\infty)$ such that

$$\frac{E^{\nu}\left[|u_n|^2\right]}{\left|E^{\nu}\left[u_n u_{n+1} u_{n+2}\right]\right|^{2/3}} \leq \gamma(\nu)$$

for $(\nu, n) \in (0, \nu_0] \times [1, n_+(\nu)]$ and

$$\limsup_{\nu \rightarrow 0} \frac{\gamma(\nu)}{\log \nu^{-1}} < \infty.$$

Thus a logarithmic divergence in ν is still acceptable. In such a case one has

$$E^{\nu}\left[u_n^2\right] \leq 2\gamma(\nu)\,\sigma^2 k_n^{-2/3}$$

for $(\nu, n) \in (0, \nu_0] \times [1, n_+(\nu)]$ and

$$\zeta_2^- \geq \frac{2}{3}$$

relative to any range of the form $[n_-(\nu), n_+(\nu)]$ with diverging $n_-(\nu)$ such that

$$\lim_{\nu \rightarrow 0} \frac{\log_2 \gamma(\nu)}{n_-(\nu)} = 0.$$

The next theorem generalizes this idea in a sort of optimal way.

Let us also remark that one can perform generalizations in other directions: for instance one can include a suitable dependence on n in the function γ, and a stronger dependence on ν for sufficiently small n. These generalizations do not seem to be motivated at present.

Theorem 31 *With $n_-(\nu) = 1$, assume (see also remark 24)*

$$\limsup_{\substack{\nu \rightarrow 0 \\ (\nu,n) \in R}} \frac{1}{\log \nu^{-1}} \log \frac{E^{\nu}\left[|u_n|^2\right]}{\left|E^{\nu}\left[u_n u_{n+1} u_{n+2}\right]\right|^{2/3}} \leq 0 \tag{27}$$

Then we have claim 30, (26) with $n_-(\nu) = 1$, and

$$\zeta_2^- \geq \frac{2}{3}$$

relative to any range of the form $[n_-(\nu), n_+(\nu)]$ satisfying (24).

Proof. We have only to check claim 30 and the conclusion on ζ_2^-.

From (27), for every $\varepsilon_0 > 0$ there is $\nu_0 > 0$ such that

$$E^\nu\left[|u_n|^2\right] \leq \nu^{-\varepsilon_0}\left|E^\nu\left[u_n u_{n+1} u_{n+2}\right]\right|^{2/3}$$

for $(\nu, n) \in (0, \nu_0] \times [1, n_+(\nu)]$, thus

$$\begin{aligned}\epsilon_n &\leq \nu^{1-\varepsilon_0} k_n^2 \left|E\left[u_n u_{n+1} u_{n+2}\right]\right|^{2/3} = 2^{-4/3}\nu^{1-\varepsilon_0} k_n^{4/3}\left|\phi_n\right|^{2/3} \\ &\leq \left(2^{-4/3}\nu^{\varepsilon_1}\right)\left(\nu^{1-\varepsilon_0-\varepsilon_1} k_{n_+(\nu)}^{4/3+\varepsilon_2}\right) k_n^{-\varepsilon_2}\left|\phi_n\right|^{2/3}\end{aligned}$$

for every $\varepsilon_1, \varepsilon_2 > 0$, and the proof of claim 30 is the same as in the previous theorem, replacing the choice of $\varepsilon_1, \varepsilon_2$ (relative to ε of that proof) by the choice of $\varepsilon_0 + \varepsilon_1, \varepsilon_2$.

About ζ_2^- we just use lemma 23. The proof is complete. ■

We can also prove a converse implication. Unfortunately we cannot express theorems 31 and 32 as an *if an only if* (similarly to lemma 23) because in both theorems we impose the assumptions on the range $[1, n_+(\nu)]$ (to start an iteration procedure) and deduce the conclusions on a smaller range $[n_-(\nu), n_+(\nu)]$ subject to property (24) (to delete minor divergences).

Theorem 32 *If*

$$\zeta_2^- \geq \frac{2}{3}$$

relative to a range $[1, n_+(\nu)]$ *then we have (26) on* $[1, n_+(\nu)]$*, and (27) on any range satisfying (24).*

Proof. Step 1 (estimates on ϵ_n). In place of claim 30 we have:

Claim 33 *there is* $\alpha \in (0,1)$ *depending only on* n_+ *and given any* $\delta > 0$ *there is* $\nu_0 > 0$ *depending only on* δ *and* n_+*, such that*

$$\epsilon_n \leq \delta \alpha^n$$

for $(\nu, n) \in (0, \nu_0] \times [1, n_+(\nu)]$.

From $\zeta_2^- \geq \frac{2}{3}$, for every $\varepsilon > 0$ there is ν_0 such that

$$\frac{1}{n}\log_2 E^\nu\left[|u_n|^2\right] \leq -\frac{2}{3} + \varepsilon$$

for $(\nu, n) \in (0, \nu_0] \times [1, n_+(\nu)]$, and thus

$$E^\nu \left[|u_n|^2 \right] \leq 2^{-n\left(\frac{2}{3}-\varepsilon\right)} = k_n^{-\frac{2}{3}+\varepsilon}.$$

Therefore

$$\epsilon_n \leq \nu k_n^2 k_n^{-\frac{2}{3}+\varepsilon} \leq \nu^{\varepsilon_1} \left(\nu^{1-\varepsilon_1} k_{n_+(\nu)}^{4/3+\varepsilon_2} \right) k_n^{-\varepsilon_2}$$

for every $\varepsilon_1, \varepsilon_2 > 0$. From the assumption on $n_+(\nu)$, if we choose $\varepsilon_1, \varepsilon_2 > 0$ such that $\frac{1-\varepsilon_1}{4/3+\varepsilon_2} = \frac{3}{4} - \varepsilon$ we get

$$\epsilon_n \leq \nu^{\varepsilon_1} k_n^{-\varepsilon_2}.$$

This proves claim 33.

Step 2 (estimate from below for $\phi_1, ..., \phi_4$). Choose δ such that $\delta\alpha \leq \frac{\sigma^2}{8}$. Let us work for $(\nu, n) \in (0, \nu_0] \times [1, n_+(\nu)]$. From (25) we have

$$\phi_1 + \epsilon_1 = \frac{\sigma^2}{4}$$

where $\epsilon_1 \leq \delta\alpha$ (claim 33). Hence

$$\phi_1 \geq \frac{\sigma^2}{4} - \delta\alpha.$$

In particular, by the smallness of δ, $\phi_1 \geq 0$. For $n = 2$, from (25) we have

$$\phi_2 + \epsilon_2 = \frac{\phi_1}{2} + \frac{\sigma^2}{4}$$

where $\epsilon_2 \leq \delta\alpha^2$ (claim 33), so

$$\phi_2 \geq \frac{\phi_1}{2} + \frac{\sigma^2}{4} - \delta\alpha^2 \geq \frac{\sigma^2}{4} - \delta\alpha^2$$

and in particular $\phi_2 \geq 0$. The same is true for ϕ_3 and ϕ_4, with α^3 and α^4, so we have proved:

$$\phi_n \geq \frac{\sigma^2}{4} - \delta\alpha^n \text{ for } n = 1, 2, 3, 4.$$

Step 3 (estimate from below for ϕ_n, $n \in [5, n_+(\nu)]$). In addition to $\delta\alpha \leq \frac{\sigma^2}{8}$ we also require that

$$\frac{\delta\sigma^{4/3}}{1-\alpha} \leq \frac{\sigma^2}{8}.$$

We prove by induction that

$$\phi_n \geq \frac{\sigma^2}{4} - \delta\left(1+\alpha+...+\alpha^n\right)$$

for $(\nu, n) \in (0, \nu_0] \times [1, n_+(\nu)]$. This will complete the proof of (26). This inequality is true for $n = 1, ..., 4$. Let us assume it is true for $n = 1, ..., k$, for some $k \geq 4$, and let us prove it for $n = k+1$. From (25) for $n = k+1$ we have

$$\phi_{k+1} + \epsilon_{k+1} = \frac{\phi_{k-1} + \phi_k}{2}$$

with $\epsilon_{k+1} \leq \delta\alpha^{k+1}$ (claim 33), so

$$\phi_{k+1} \geq \frac{\phi_{k-1} + \phi_k}{2} - \delta\alpha^{k+1}$$

$$\geq \frac{\sigma^2}{4} - \delta\frac{\left(1+\alpha+...+\alpha^{k-1}\right) + \left(1+\alpha+...+\alpha^k\right)}{2} - \delta\alpha^{k+1}$$
$$\geq \frac{\sigma^2}{4} - \delta\left(1+\alpha+...+\alpha^k\right) - \delta\alpha^{k+1}.$$

The proof of (26) is complete.

Step 4. Finally, to obtain (27) it is sufficient to use lemma 23. The proof is complete. ■

We give now a criterium for the Kolmogorov scaling $\zeta_2 = \frac{2}{3}$, following lemma 23. It is again based on an unproved assumption. However, while on the basis of numerical simulations of us and other authors we strongly believe that the assumptions of theorem 31 are satisfied, so $\zeta_2^- \geq \frac{2}{3}$, we are still unsure about the assumptions and the conclusion of the next theorem.

Theorem 34 *With $n_-(\nu) = 1$, assume (see also remark 25)*

$$\lim_{\substack{\nu\to 0 \\ (\nu,n)\in R}} \frac{1}{\log \nu^{-1}} \log \frac{E^\nu\left[|u_n|^2\right]}{\left|E^\nu\left[u_n u_{n+1} u_{n+2}\right]\right|^{2/3}} = 0. \tag{28}$$

Then

$$\zeta_2 = \frac{2}{3}$$

relative to any range satisfying (24).

Proof. From theorem 31 we already know $\zeta_2^- \geq \frac{2}{3}$, but mainly we know that (26) on $[1, n_+(\nu)]$ is true. Thus we may apply lemma 23. The proof is complete. ■

Similarly, we also have:

Theorem 35 *With $n_-(\nu) = 1$, assume*

$$\zeta_2 = \frac{2}{3}.$$

Then we have (28) on any range satisfying (24).

5.1 Numerical results

In this section we simulate system(2) in the particular case when the noise satisfies condition (22) with $\sigma = 1$. We are interested in mean values in the stationary regime: hence we start with the trivial initial condition $u_n(0) = 0$ for every n and we compute time means after a certain transient period. The relation between time averages and probabilistic expectation, namely the ergodicity of the invariant measure, is not studied rigorously in this paper and will be the object of future researches; but the experience with 2D Navier-Stokes equations suggests that we should expect ergodicity (see in particular [15]).

In our simulations we observe the time evolution of local-in-time averages of some observable like $|u_n|^2$ for small and large n. Then we decide by explicit observation when the stationary regime is reached. However, we could have lost events which appear only with exponentially small probability. For this and other reasons, we do not declare the results of these simulations as conclusive but only as a first rough indication.

Of course we have to cut the infinite dimensional system. Hence, given N, we impose the boundary conditions

$$u_{N+1}(t) = u_{N+2}(t) = 0.$$

The choice of N, for a given $\nu > 0$, is of a number of the order

$$N \sim \log_2 \nu^{-1}$$

(thus since we take ν of the form 10^{-K}, N will be roughly $3.3 \cdot K$). This choice looks natural since we want to observe the range $[1, n_+(\nu)]$ with $n_+(\nu) <<$

$\frac{3}{4}\log_2 \nu^{-1}$, and by rough arguments it seems that the dissipation in the range $\left[\frac{3}{4}-\varepsilon, 1\right]\log_2 \nu^{-1}$ is exhaustive.

Computationally, we have used fourth order Runge-Kutta explicit integration with a time step roughly of size ν. Thus the global evolution time we could check was different, depending on ν. For this reason the reliability of the results of our simulations may decrease with ν.

For a given $\nu > 0$, we plot the values of $\frac{E^{\nu}\left[|u_n|^2\right]}{|E^{\nu}[u_n u_{n+1} u_{n+2}]|^{2/3}}$ in logarithmic scale against n. The purpose is to explore the validity of the conditions of theorems 31 and 34. Having in mind remarks 24 and 25, we explore whether the following upper inequality or even both ones

$$\frac{C_1}{\log \nu^{-1}} \leq \frac{E^{\nu}\left[|u_n|^2\right]}{|E^{\nu}\left[u_n u_{n+1} u_{n+2}\right]|^{2/3}} \leq C_2 \log \nu^{-1}$$

could be satisfies for $(\nu, n) \in (0, \overline{\nu}] \times [n_-(\nu), n_+(\nu)]$ with some $\overline{\nu} > 0$, $C_1, C_2 > 0$. In figure 1 we show the result for

$$\nu = 10^{-6}, 10^{-8}, 10^{-10}, 10^{-12}.$$

Standing that no conclusion can be drawn from four values, the indication is that the upper bound, related to theorem 31, should be true. There is a little uncertainty about the validity of the upper bound for very small n, but if more careful numerics should confirm that (just) for such n the upper bound increases strongly than $C_2 \log \nu^{-1}$, one still could modify theorem 31 (we have not described all possible generalizations, but the iterative procedure in the proof of such theorems can start under more general assumptions, since for small n the prefactor of $E^{\nu}\left[|u_n|^2\right]$ in the balance relations is extremely small). In conclusion, we think that our simulations and theorems are a strong support for the result

$$\zeta_2^- \geq \frac{2}{3}.$$

On the other side, it is more critical to draw conclusions about the lower bound. On one side, the numerical values of the lower bounds in each plot is encouraging, around 1 or at most 1/2, so one could think that also the assumptions of theorem 34 are satisfied, and $\zeta_2 = \frac{2}{3}$. But the *decreasing* in n shape of all curves is a major feature that alarm us. Comparing just the four plots, the (approximate) slope of the curves is smaller for smaller values of ν. So we cannot decide between the following two possibilities, that should be observed in difficult experiments with much smaller values of ν: i) either this slope goes to zero (as ν decreases), the lower bound is of order $\frac{C_1}{\log \nu^{-1}}$ and $\zeta_2 = \frac{2}{3}$; ii) or the slope does not go to zero, and we start to appreciate that the lower bound is an infinitesimal of higher order than those admissible for theorem 34. There are also other possibilities, including: iii) that K41 scaling is true on a much smaller range $\left[1, n_+^0(\nu)\right]$, with $n_+^0(\nu) << n_+(\nu)$, while on $\left[n_+^0(\nu), n_+(\nu)\right]$ we observe other exponents, slightly larger than 2/3.

In any case, the slope is an indication that some form of correction to K41 scaling exists. Perhaps it cannot be captured by the definition of ζ_2 chosen here.

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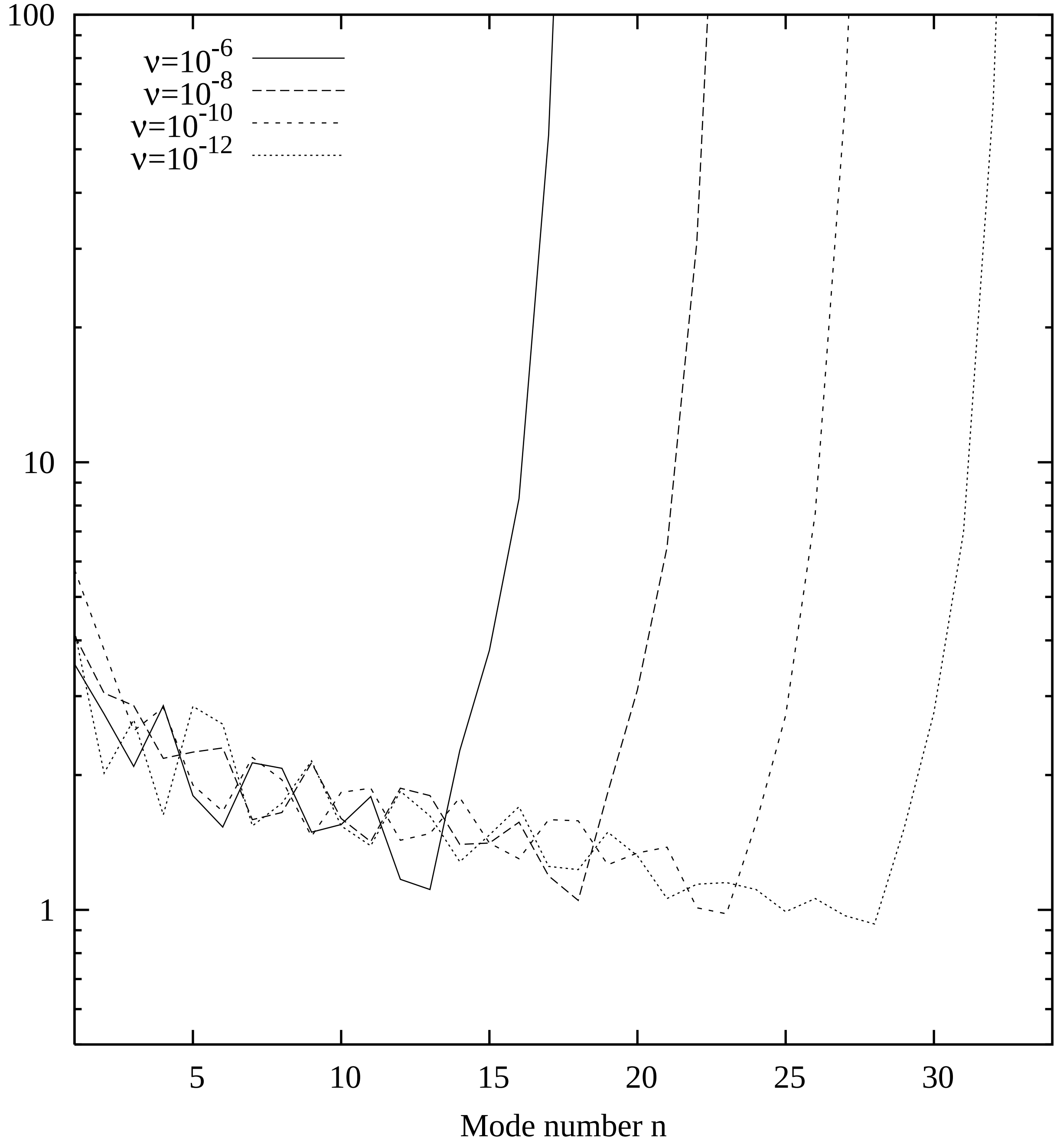

Figure 1: Values of $\frac{E^{\nu}\left[|u_n|^2\right]}{\left|E^{\nu}[u_n u_{n+1} u_{n+2}]\right|^{2/3}}$ in logarithmic scale *vs.* n for $\nu = 10^{-6}$, 10^{-8}, 10^{-10}, 10^{-12}.

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