# Regularity and asymptotic behaviour of the solutions of the Navier-Stokes equations with diffusion.

Hakima BESSAIH

Scuola Normale Superiore

Piazza dei Cavalieri,7

56126 Pisa (Italy)

**Abstract**

In this paper we consider the motion of a continuous medium consisting of two components, for example pure and salt water, with a diffusion effect obeying Fick's law. We prove, for small data and external force, the regularity of the solution in the Hilbert space $H^k$, with $k \geq 2$, and the exponential asymptotic convergence to the solutions of the homogeneous Navier-Stokes equations.

## 1 Introduction.

In this paper we consider the motion of a viscous fluid consisting of two components, for instance, saturated salt water and water. The equations of the model are obtained, for example in [5]. Let us give a brief sketch. Let $\rho_1$, $\rho_2$ be the characteristic densities (constant) of the two components, $v^1(t,x)$ and $v^2(t,x)$ their velocities and $e(t,x)$, $d(t,x)$ the mass and volume concentration of the first fluid. The mean density of the mixture is

$$\rho(t,x) = d(t,x)\rho_1(t,x) + (1 - d(t,x))\rho_2(t,x)$$

and we introduce the mean-volume and mean-mass velocities

$$v(t,x) = d(t,x)v^1(t,x) + (1 - d(t,x))v^2(t,x),$$

$$w(t,x) = e(t,x)v^1(t,x) + (1-e(t,x))v^2(t,x).$$

Then the equations of motion are given by

$$\begin{cases} \rho(\dot{w} + (w.\nabla)w - f) - \mu\Delta w - (\mu+\mu')\nabla \mathrm{div} w = -\nabla\pi, \\ \mathrm{div} v = 0, \\ \dot{\rho} + \mathrm{div}(\rho w) = 0. \end{cases} \tag{1.1}$$

On the other hand, Fick's diffusion law gives

$$w = v - \lambda\rho^{-1}\nabla\rho.$$

By eliminating $w$ in the preceeding equation one gets, after some calculations,

$$\begin{cases} \rho(\dot{v} + (v.\nabla)v) - \mu\Delta v - \lambda[(v.\nabla)\nabla\rho + (\nabla\rho.\nabla)v] & \\ +\frac{\lambda^2}{\rho}[(\nabla\rho.\nabla)\nabla\rho - \frac{1}{\rho}(\nabla\rho.\nabla\rho)\nabla\rho + \Delta\rho\nabla\rho] = & \\ -\nabla p + \rho f & \text{in } Q_T, \\ \dot{\rho} + v.\nabla\rho - \lambda\Delta\rho = 0 & \text{in } Q_T, \\ \mathrm{div} v = 0 & \text{in } Q_T, \\ v = 0 & \text{on } \Sigma_T, \\ \frac{\partial\rho}{\partial n} = 0 & \text{on } \Sigma_T, \\ v|_{t=0} = v_0(x) & \text{in } \Omega, \\ \rho|_{t=0} = \rho_0(x) & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $\Omega$ is a bounded domain in $R^3$ locally situated on one side of its boundary $\Gamma$, regular manifold, $Q_T = (0,T) \times \Omega$, $\Sigma_T = (0,T) \times \Gamma$ and $0 < T \leq \infty$. Here $\rho$ is the density of the fluid, $v$ the velocity, $f$ the external (assigned) force and $p = p(\rho)$ the pressure which is assumed to be a known function of the density. $\mu$ and $\lambda$ are respectively the viscosity and the diffusion coefficient. Finally $v_0(x)$ and $\rho_0(x)$ are the initial velocity and density respectively. The significance of the boundary conditions is that the fluid is isolated, i.e. there is no flux through the boundary.

In (1.2) and in what follows, we will use the notations

$$\dot{v} = \frac{\partial v}{\partial t}, \quad \dot{\rho} = \frac{\partial \rho}{\partial t}, \quad (v.\nabla)v = \sum_{i=1}^{3} v_i \frac{\partial v}{\partial x_i}.$$

Let us introduce the following functional spaces:

$$H_N^k = \left\{ \sigma \in H^k(\Omega), \frac{\partial \sigma}{\partial n} = 0 \text{ on } \Gamma \text{ and } \int_\Omega \sigma(x)dx = 0 \right\},\ k \geq 2.$$

$$\mathcal{V} = \left\{ v \in C_0^\infty(\Omega), \text{div} v = 0 \text{ in } \Omega \right\}.$$

$$H = \left\{ v \in L^2(\Omega), \text{div} v = 0 \text{ in } \Omega,\ v.n = 0 \text{ on } \Gamma \right\}.$$

$$V = \left\{ v \in H_0^1(\Omega),\ \text{div} v = 0 \text{ on } \Omega \right\}.$$

H and V are the closures of $\mathcal{V}$ in $L^2(\Omega)$ and $H_0^1(\Omega)$ respectively.
We will denote the norm in $H^k(\Omega)$ (the usual Sobolev space) by $\| \ . \ \|_k$ ($\| u \|_k^2 = \sum_{l=0}^k \| D^l u \|_0^2$ where $\| D^l u \|_0^2 = \sum_{|\alpha|=l} \| D^\alpha u \|_0^2$) for $k \in N$, the norm in $L^p((0,T); H^k(\Omega))$ by $\| \ . \ \|_{p,k}$ for $0 \leq p \leq \infty$, $k \in N$ and $0 < T \leq \infty$. The norm in $L^\infty((0,T); X)$ and in $C((0,T); X)$ are denoted in the same way, where $C((0,T); X)$ is the space of continuous and bounded functions from $[0,T]$ to $X$.
Moreover, it is useful to remark that the norms $\| \sigma \|_k$ and $\| D^k \sigma \|_0$ are equivalent in $H_N^k$ and that $\| v \|_k$, $\| D^k v \|_0$ are equivalent in $V \cap H^k(\Omega)$ (see Necãs [6]).
We set

$$m = \inf_{x \in \Omega} \rho_0(x), \tag{1.3}$$

$$M = \sup_{x \in \Omega} \rho_0(x), \tag{1.4}$$

and

$$\overline{\rho} = \frac{1}{|\Omega|} \int_\Omega \rho_0(x)dx. \tag{1.5}$$

Further

$$\rho(t,x) = \sigma(t,x) + \overline{\rho}. \tag{1.6}$$

We assume that $m > 0$.
Obviously one has

$$m \leq \overline{\rho} \leq M. \tag{1.7}$$

We denote by $k_1$, $k_2$,..., c, $C_0$, $C_1$,..., $K_1$, $K_2$, $K_3$ positive constants depending at most on $\Omega$ and on the parameters $\mu$, $\lambda$, $m$, M, $\overline{\rho}$. For convenience, we sometimes denote differents constants by the same symbol c, even in the same equation.

In [5] *Kazhikov* and *Smagulov* consider the simplified system obtained from $(1.2)_1$, by omitting the term containing $\lambda^2$. Moreover, they assume that

$$0 < \lambda < \frac{2\mu}{M-m}. \tag{1.8}$$

Under these conditions *Kazhikov* and *Smagulov* state the existence of a local solution in time. They also prove in the two dimensional case, that the solution is global in time.

Local existence in the general case (i.e. with the $\lambda^2$-term), without assuming (1.6) was proved for inviscid fluids ($\mu = 0$) in [3]. In [1], *Beirão da Veiga* considers the full equations (viscous fluids), without the restriction (1.6), and proves (i)the existence of a (unique) local solution, (ii) the existence of a global solution in time for small initial velocities and external forces, (iii) the exponential decay (when $t \to \infty$) of the solution $(\rho, v)$ to the equilibrium solution $(\overline{\rho}, 0)$, if $f = 0$. The proof, specific for viscous flows, relies essentially on a balance estimate obtained by taking the inner product in $H$ of the projection of the main equation $(1.2)_1$ into $H$, with $\dot{v} + \epsilon A v$ (where $Av = Pr\Delta$), and by choosing $\epsilon > 0$ in a convenient way.

In the present work, we are concerned with the regularity and the asymptotic behaviour of the solutions of the problem (1.2).

# 2 Main results.

The Theorem 2.1 below generalizes to an arbitrary $k \geq 2$ results proved in reference [1] for $k = 0$. The particular case $k = 1$ requires some modifications and, for brevity, we will not present it here. Hence we will assume in the sequel that $k \geq 2$ be a fixed integer.

**Theorem 2.1** *Let $v_0 \in V \cap H^{k+2}(\Omega)$, $\rho_0 \in H^{k+2}_N$, $f \in L^\infty((0,T); H^{k+1}(\Omega))$, $\dot{f} \in L^\infty((0,T); H^{k-1}(\Omega))$. Then exists $T_1 \in ]0,T]$ such that problem (1.2) is uniquely solvable in $Q_{T_1}$. Moreover*

$$v \in C((0,T_1); V \cap H^{k+2}(\Omega)) \cap L^2((0,T_1); H^{k+4}(\Omega)),$$

$$\dot{v} \in L^\infty((0,T_1); H^k(\Omega)) \cap L^2((0,T_1); H^{k+2}(\Omega)),$$

$$\rho \in C((0,T_1);H_N^{k+2}(\Omega)) \cap L^2((0,T_1);H_N^{k+4}(\Omega)),$$
$$\dot{\rho} \in L^\infty((0,T_1);H^k(\Omega)) \cap L^2((0,T_1);H^{k+2}(\Omega)),$$

*and*

$$m \leq \rho(t,x) \leq M.$$

*Moreover, there exist positive constants* $k_1$, $k_2$ *(depending at most on* $\Omega, \mu, \lambda$ *and on the mean density* $\overline{\rho}$*) such that if*

$$\| v_0 \|_{k+2}^2 + \| \rho_0 - \overline{\rho} \|_{k+2}^2 \leq k_1, \tag{2.1}$$

*and*

$$\| f \|_{L^\infty(0,\infty;H^{k+1})}^4 + \| \dot{f} \|_{L^\infty(0,\infty;H^{k-1})}^4 \leq k_2, \tag{2.2}$$

*then the solution is global in time. Moreover, if* $f = 0$ *then the solution* $(\rho, v)$ *decays exponentially to the equilibrium solution* $(\overline{\rho}, 0)$*, ie*

$$\| v \|_{k+2}^2 + \| \rho - \overline{\rho} \|_{k+2}^2 \leq k_3 \left( \| v_0 \|_{k+2}^2 + \| \rho_0 - \overline{\rho} \|_{k+2}^2 \right) \mathrm{e}^{-k_4 t}, \tag{2.3}$$

*for every* $t \geq 0$*.*

The next theorem generalizes the Theorem 1.4 in reference [2](see Theorem 5.2 below).

**Theorem 2.2** *There exist positive constants* $k_5$ *and* $k_6$ *such that, if (2.1) and (2.2) hold and moreover the conditions*

$$\left( \| v_0 \|_V^2 + \| \Delta(\rho_0 - \overline{\rho}) \|_0^2 \right)^2 \leq k_5, \tag{2.4}$$

*and*

$$\| f \|_{L^\infty(0,\infty;L^2)}^2 \leq k_6, \tag{2.5}$$

*are satisfied, then for each* $r = (k+2) - \theta k,\ \theta \in ]0,1[$*, there exist positive constants* $k_7$ *and* $k_8$ *such that*

$$\| (\rho(t) - \overline{\rho}) \|_r \leq k_7 \mathrm{e}^{-k_8 t} \| (\rho_0 - \overline{\rho}) \|_2^\theta . \tag{2.6}$$

*If moreover* $(\overline{v}, \overline{\sigma})$ *is another solution of (1.2) with initial data* $(\bar{v}_0, \bar{\sigma}_0)$ *satisfying (2.1) and (2.4) then for each* $r_1 = (1-\theta)(k+2) + s\theta,\ \theta \in ]0,1[,\ s \in [0,1[$*, there exist positive constants* $k_9$ *and* $k_{10}$ *such that*

$$\| v(t) - \overline{v}(t) \|_{r_1} \leq k_9 \mathrm{e}^{-k_{10} t} \quad \forall t \geq 0. \tag{2.7}$$

*In particular, if* $(w, 0)$ *is a solution of the homogeneous Navier-Stokes equations*

$$\begin{cases} \overline{\rho}(\dot{w} + (w.\nabla)w) - \mu\Delta w = -\nabla\pi + \overline{\rho}f & \text{in } Q_T, \\ \mathrm{div} w = 0 & \text{in } Q_T, \\ w = 0 & \text{on } \Sigma_T, \\ w|_{t=0} = w_0(x) & \text{in } \Omega. \end{cases} \tag{2.8}$$

*with initial data* $w_0$ *satisfying* $\| w_0 \|_V^4 \leq k_5$*, then*

$$\| v(t) - w(t) \|_{r_1} + \| \rho(t) - \overline{\rho} \|_r \leq k_{11}\mathrm{e}^{-k_{12}t}, \tag{2.9}$$

*for each* $t \geq 0$*.*

# 3 The linearized equations.

We start by proving the following theorem:

**Theorem 3.1** *Let* $\tilde{\rho}(t, x)$ *be a measurable function satisfying*

$$0 < m \leq \tilde{\rho}(t, x) \leq M, \quad a.e. in \quad Q_T, \tag{3.1}$$

$$\tilde{\rho} - \overline{\rho} \in C^0(0, T; H^{k+2}(\Omega)) \cap L^2(0, T; H^{k+2}(\Omega)),$$

$$\dot{\tilde{\rho}} \in L^\infty(0, T; H^k(\Omega)) \cap L^2(0, T; H^{k+2}(\Omega)),$$

*and let* $F \in L^2(0, T; H^{k+2}(\Omega))$*,* $\dot{F} \in L^2(0, T; H^{k-1}(\Omega))$ *and* $v_0(x) \in V \cap H^{k+2}$*. Then there exists a (unique) strong solution* $v$ *of the problem*

$$\begin{cases} \tilde{\rho}\dot{v} - \mu\Delta v = F & \text{in } Q_T, \\ \mathrm{div} v = 0 & \text{in } Q_T, \\ v = 0 & \text{on } \Sigma_T, \\ v|_{t=0} = v_0(x) & \text{in } \Omega. \end{cases} \tag{3.2}$$

*Moreover*

$$v \in C((0, T); V \cap H^{k+2}(\Omega)) \cap L^2((0, T); V \cap H^{k+4}(\Omega)),$$

$$\dot{v} \in L^\infty((0, T); H^k(\Omega)) \cap L^2((0, T); H^{k+2}(\Omega)),$$

*and there exists positive constants* $C_0, C_1, C_2$ *for which the following estimate holds*

$$\begin{aligned}
&\| \dot{v} \|_{\infty,k}^2 + \| v \|_{\infty,k+2}^2 + \| \dot{v} \|_{2,k+2}^2 + \| v \|_{2,k+4}^2 \leq \\
&C_0(\| v_0 \|_{k+2}^2 \exp C_1 T \| \dot{\tilde{\sigma}} \|_{\infty,k}^2 + (\| F \|_{2,k+1}^2 \| \tilde{\sigma} \|_{\infty,k+1}^2 + \| F \|_{2,k+2}^2) \\
&\times (1 + \exp C_1 T \| \dot{\tilde{\sigma}} \|_{\infty,k}^2) + [T(\| \tilde{\sigma} \|_{\infty,k+2}^2 + \| \tilde{\sigma} \|_{\infty,k+1}^4) \\
&\times (1 + \exp C_1 T \| \dot{\tilde{\sigma}} \|_{\infty,k}^2) + 1][\| \dot{v}_0 \|_k^2 + \| \dot{F} \|_{2,k-1}^2 \\
&+ \| F \|_{2,k+2}^2 (1 + \| \tilde{\sigma} \|_{\infty,k+1}^2 + \| \tilde{\sigma} \|_{\infty,k+1}^4) \\
&+ \| F \|_{2,k+1}^2 (\| \tilde{\sigma} \|_{\infty,k+1}^2 + \| \tilde{\sigma} \|_{\infty,k+1}^4 + \| \tilde{\sigma} \|_{\infty,k+1}^6) \\
&+ \| \dot{F} \|_{2,k}^2 (\| \tilde{\sigma} \|_{\infty,k+1}^2 + \| \tilde{\sigma} \|_{\infty,k+1}^4)] \\
&\exp C_2[\| \dot{\tilde{\sigma}} \|_{2,k+1}^2 (1 + \| \tilde{\sigma} \|_{\infty,k+1}^2 + \| \tilde{\sigma} \|_{\infty,k+1}^4) \\
&+T(\| \tilde{\sigma} \|_{\infty,k+1}^2 + \| \tilde{\sigma} \|_{\infty,k+1}^4 + \| \tilde{\sigma} \|_{\infty,k+1}^8 \\
&+ \| \tilde{\sigma} \|_{\infty,k+1}^2 \| \tilde{\sigma} \|_{\infty,k+2}^2 + \| \tilde{\sigma} \|_{\infty,k+1}^4 \| \tilde{\sigma} \|_{\infty,k+2}^2)].
\end{aligned} \tag{3.3}$$

Before making the Proof of the above theorem, we need some lemmas.

**Lemma 3.2** *One has the following estimate*

$$\begin{aligned}
\| D^{k+2}\dot{v} \|_0^2 \quad &+ \quad \| D^{k+2}\Delta v \|_0^2 \leq C_3(\| F \|_{k+2}^2 + \| \tilde{\sigma} \|_{k+2}^2 \| \dot{v} \|_k^2 \\
&+ \quad \| \tilde{\sigma} \|_{k+1}^4 \| \dot{v} \|_k^2 + \| \tilde{\sigma} \|_{k+1}^2 \| F \|_{k+1}^2).
\end{aligned} \tag{3.4}$$

**Proof**: We apply the operator $D^k$ to equation $(3.2)_1$ we obtain

$$\begin{cases}
D^k(\tilde{\rho}\dot{v}) - \mu D^k \Delta v = D^k F & \text{in } Q_T, \\
\text{div} v = 0 & \text{in } Q_T, \\
v = 0 & \text{on } \Sigma_T, \\
D^k v|_{t=0} = D^k v_0(x) & \text{in } \Omega.
\end{cases} \tag{3.5}$$

We take the inner product of $(3.5)_1$ with $(D^k\dot{v} - \epsilon_0 D^k \Delta v)$, $\epsilon_0$ being an arbitrary positive constant. We integrate on $\Omega$ and we obtain

$$\int_\Omega D^k(\tilde{\rho}\dot{v}) D^k \dot{v} + \mu\epsilon_0 \| D^k \Delta v \|_0^2 = \epsilon_0 \int_\Omega D^k(\tilde{\rho}\dot{v}) D^k \Delta v$$

$$+\mu \int_\Omega D^k \dot{v} D^k \Delta v + \int_\Omega D^k F D^k \dot{v} - \epsilon_0 \int_\Omega D^k F D^k \Delta v,$$

but one has

$$D^k(\tilde{\rho}\dot{v}) = D^k \tilde{\rho} \dot{v} + \tilde{\rho} D^k \dot{v} + D^{k-1}(\tilde{\rho}\dot{v}).$$

Using (3.1) to estimate $\tilde{\rho}$ in the second term at the right hand side of the above equality, one gets

$$\begin{aligned} m \parallel D^k \dot{v} \parallel_0^2 &+ \mu\epsilon_0 \parallel D^k \Delta v \parallel_0^2 \leq \parallel D^k \tilde{\sigma} \parallel_0 \parallel \dot{v} \parallel_2 \parallel D^k \dot{v} \parallel_0 \\ &+ \parallel D^{k-1}(\tilde{\rho}\dot{v}) \parallel_0 \parallel D^k \dot{v} \parallel_0 + \epsilon_0 \parallel D^k(\tilde{\rho}\dot{v}) \parallel_0 \parallel D^k \Delta v \parallel_0 \\ &+ \mu \parallel D^k \Delta v \parallel_0 \parallel D^k \dot{v} \parallel_0 + \parallel D^k F \parallel_0 \parallel D^k \dot{v} \parallel_0 \\ &+ \parallel D^k F \parallel_0 \parallel D^k \Delta v \parallel_0 . \end{aligned}$$

On the other hand (see the appendix Lemma 6.2), one has

$$\parallel D^{k-1}(\tilde{\rho}\dot{v}) \parallel_0 \leq c \parallel D\tilde{\sigma} \parallel_\infty \parallel \dot{v} \parallel_{k-2} + \parallel D\tilde{\sigma} \parallel_{k-2} \parallel \dot{v} \parallel_\infty ,$$

and

$$\parallel D^k(\tilde{\rho}\dot{v}) \parallel_0 \leq c \parallel D\tilde{\sigma} \parallel_\infty \parallel \dot{v} \parallel_{k-1} + \parallel D\tilde{\sigma} \parallel_{k-1} \parallel \dot{v} \parallel_\infty .$$

Since $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ and

$$\mu \parallel D^k \Delta v \parallel_0 \parallel D^k \dot{v} \parallel_0 \leq \frac{\mu\epsilon_0}{4} \parallel D^k \Delta v \parallel_0^2 + \frac{\mu}{\epsilon_0} \parallel D^k \dot{v} \parallel_0^2 ,$$

and by choosing $\epsilon_0 = \frac{8\mu}{m}$, one obtains the following estimate

$$\begin{aligned} \parallel D^k \dot{v} \parallel_0^2 + \parallel D^k \Delta v \parallel_0^2 &\leq C_4 (\parallel \tilde{\sigma} \parallel_k^2 \parallel \dot{v} \parallel_2^2 \\ &+ \parallel \tilde{\sigma} \parallel_3^2 \parallel \dot{v} \parallel_{k-1}^2 + \parallel F \parallel_k^2). \end{aligned} \tag{3.6}$$

Now we write (3.6) at the order $(k+1)$ and $(k+2)$, we obtain

$$\begin{aligned} \parallel D^{k+1} \dot{v} \parallel_0^2 + \parallel D^{k+1} \Delta v \parallel_0^2 &\leq C_4 (\parallel \tilde{\sigma} \parallel_{k+1}^2 \parallel \dot{v} \parallel_2^2 \\ &+ \parallel \tilde{\sigma} \parallel_3^2 \parallel \dot{v} \parallel_k^2 + \parallel F \parallel_{k+1}^2), \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \parallel D^{k+2} \dot{v} \parallel_0^2 + \parallel D^{k+2} \Delta v \parallel_0^2 &\leq C_4 (\parallel \tilde{\sigma} \parallel_{k+2}^2 \parallel \dot{v} \parallel_2^2 \\ &+ \parallel \tilde{\sigma} \parallel_3^2 \parallel \dot{v} \parallel_{k+1}^2 + \parallel F \parallel_{k+2}^2). \end{aligned} \tag{3.8}$$

Finally we use (3.7) to estimate $\parallel \dot{v} \parallel_{k+1}^2$ in (3.8), and the fact that $k \geq 2$, and we obtain (3.4).□

**Lemma 3.3** *One has*

$$\frac{d}{dt}\int_\Omega \tilde\rho |D^{k+2}v|^2 + \| D^{k+2}\nabla v \|_0^2 \leq C_5(\| D^{k+2}\nabla v \|_1^2$$
$$+ \| D^{k+2}F \|_0^2 + \| \dot{\tilde\sigma} \|_k^2 \| v \|_{k+2}^2 + \| \tilde\sigma \|_{k+2}^2 \| \dot v \|_k^2). \qquad (3.9)$$

**Proof**: We take the inner product of the equation $(3.5)_1$(at the order $k+2$) with $D^{k+2}v$, we integrate on $\Omega$ and we obtain

$$\int_\Omega \tilde\rho D^{k+2}\dot v . D^{k+2}v - \mu \int_\Omega D^{k+2}\Delta v . D^{k+2}v =$$
$$\int_\Omega (D^{k+2}F - D^{k+2}\tilde\rho . \dot v - D^{k+1}(\tilde\rho \dot v)) . D^{k+2}v.$$

By integrating by parts the second term on the left hand side, one has

$$\mu \int_\Omega D^{k+2}\Delta v . D^{k+2}v + \mu \int_\Omega |D^{k+2}\nabla v|^2 = < \gamma_n D^{k+2}\nabla v, \gamma_0 D^{k+2}v >_{\frac{-1}{2},\frac{1}{2}} .$$

In addition

$$\int_\Omega \tilde\rho D^{k+2}\dot v . D^{k+2}v = \frac{1}{2}\frac{d}{dt}\int_\Omega \tilde\rho |D^{k+2}v|^2 - \frac{1}{2}\int_\Omega \dot{\tilde\sigma} |D^{k+2}v|^2,$$

and

$$\| D^{k+2}\nabla v \|_1^2 \| D^{k+2}v \|_0^2 \leq \frac{3\mu}{4} \| D^{k+2}\nabla v \|_0^2 + \frac{C}{\mu} \| D^{k+2}\nabla v \|_1^2 .$$

since

$$| < \gamma_n D^{k+2}\nabla v, \gamma_0 D^{k+2}v >_{\frac{-1}{2},\frac{1}{2}} | \leq C \| D^{k+2}\nabla v \|_1 \| D^{k+2}v \|_1,$$

and $\| D^{k+2}v \|_1$ and $\| D^{k+2}\nabla v \|_0$ are equivalent norms in $H^{k+3}(\Omega)$, and using $k \geq 2$ one obtains (3.9).□

**Lemma 3.4** *One has*

$$\frac{d}{dt}\int_\Omega \tilde\rho |D^k \dot v|^2 + \| D^k \nabla \dot v \|_0^2 \leq C_6[\| \dot F \|_{k-1}^2 + \| D^k \nabla \dot v \|_1^2$$
$$+ \| \dot F \|_k^2 (\| \tilde\sigma \|_{k+1}^2 + \| \tilde\sigma \|_{k+1}^4) + \| F \|_{k+2}^2 (\| \tilde\sigma \|_{k+1}^2 + \| \tilde\sigma \|_{k+1}^4)$$
$$+ \| F \|_{k+1}^2 (\| \tilde\sigma \|_{k+1}^4 + \| \tilde\sigma \|_{k+1}^6)$$
$$+ \| \dot v \|_k^2 (\| \dot{\tilde\sigma} \|_{k+1}^2 + \| \tilde\sigma \|_{k+1}^2 + \| \dot{\tilde\sigma} \|_{k+1}^2 \| \tilde\sigma \|_{k+1}^2$$
$$+ \| \dot{\tilde\sigma} \|_{k+1}^2 \| \tilde\sigma \|_{k+1}^4 + \| \tilde\sigma \|_{k+1}^6 + \| \tilde\sigma \|_{k+1}^8$$
$$+ \| \tilde\sigma \|_{k+1}^2 \| \tilde\sigma \|_{k+2}^2 + \| \tilde\sigma \|_{k+1}^4 \| \tilde\sigma \|_{k+2}^2)]. \qquad (3.10)$$

**Proof**: We take the time derivative of the problem (3.5). We set $u = \dot{v}$ and obtain the following problem

$$\begin{cases} \tilde{\rho} D^k \dot{u} - \mu D^k \Delta u = +D^k \dot{F} - D^k(\dot{\tilde{\rho}} u) & \\ \qquad\qquad\qquad -D^k \tilde{\rho} \dot{u} - D^{k-1}(\tilde{\rho} \dot{u}) & \text{in } Q_T, \\ \operatorname{div} u = 0 & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ D^k u|_{t=0} = D^k u_0(x) & \text{in } \Omega. \end{cases} \tag{3.11}$$

We take the inner product of $(3.11)_1$ with $D^k u$ and we make the same calculations as before, we obtain

$$\begin{aligned} &\frac{d}{dt} \int_\Omega \tilde{\rho} |D^k u|^2 + \| D^k \nabla u \|_0^2 \leq C_7 (\| D^k \nabla u \|_1^2 \\ &+ \| D^k \dot{F} \|_{-1}^2 + \| \dot{\tilde{\sigma}} \|_2^2 \| u \|_k^2 + \| \tilde{\sigma} \|_{k+1}^2 \| \dot{u} \|_0^2 \\ &+ \| \dot{\tilde{\sigma}} \|_k^2 \| u \|_k^2 + \| \tilde{\sigma} \|_3^2 \| \dot{u} \|_{k-2}^2 + \| \tilde{\sigma} \|_{k-1}^2 \| \dot{u} \|_2^2). \end{aligned}$$

Moreover

$$\dot{u} = \frac{1}{\tilde{\rho}} \left( \mu \Delta u + \dot{F} - \dot{\tilde{\sigma}} u \right).$$

Hence estimating $\dot{u}$, using (3.4) and finally returning to $v$, we obtain (3.10).□

**Proof of Theorem 3.1**: Now we are able to proof the Theorem 3.1. Note that the existence of the solution was proved by Beirão da Veiga [1], we will only prove the estimate (3.3).

By adding (3.4) and (3.10) multiplied respectively by $2C_6$ and 1, so that $\| D^k \nabla \dot{v} \|_1^2$ will be eliminated in the right hand side of (3.10), one gets

$$\begin{aligned} &\frac{d}{dt} \int_\Omega \tilde{\rho} |D^k \dot{v}|^2 + \| D^{k+2} \dot{v} \|_0^2 + \| D^{k+2} \Delta v \|_0^2 \leq C_7 [\| \dot{F} \|_{k-1}^2 + \| F \|_{k+2}^2 \\ &+ \| \dot{F} \|_k^2 (\| \tilde{\sigma} \|_{k+1}^2 + \| \tilde{\sigma} \|_{k+1}^4) + \| F \|_{k+1}^2 (\| \tilde{\sigma} \|_{k+1}^2 + \| \tilde{\sigma} \|_{k+1}^4 \\ &+ \| \tilde{\sigma} \|_{k+1}^6) + \| F \|_{k+2}^2 (\| \tilde{\sigma} \|_{k+1}^2 + \| \tilde{\sigma} \|_{k+1}^4) \\ &+ \| \dot{v} \|_k^2 (\| \dot{\tilde{\sigma}} \|_{k+1}^2 + \| \tilde{\sigma} \|_{k+1}^2 + \| \tilde{\sigma} \|_{k+2}^2 + \| \tilde{\sigma} \|_{k+1}^4 \\ &+ \| \dot{\tilde{\sigma}} \|_{k+1}^2 \| \tilde{\sigma} \|_{k+1}^2 + \| \dot{\tilde{\sigma}} \|_{k+1}^2 \| \tilde{\sigma} \|_{k+1}^4 + \| \tilde{\sigma} \|_{k+1}^6 \\ &+ \| \tilde{\sigma} \|_{k+1}^8 + \| \tilde{\sigma} \|_{k+1}^2 \| \tilde{\sigma} \|_{k+2}^2 + \| \tilde{\sigma} \|_{k+1}^4 \| \tilde{\sigma} \|_{k+2}^2)]. \end{aligned} \tag{3.12}$$

By Gronwall's Lemma one obtains

$$\| D^k \dot{v} \|_{\infty,0} \quad \leq \quad C_8 [(\| \dot{v}_0 \|_k^2 + \| \dot{F} \|_{2,k-1}^2 + \| F \|_{2,k+2}^2$$

$$
\begin{aligned}
&+ \| \dot{F} \|_{2,k}^2 \, (\| \tilde{\sigma} \|_{\infty,k+1}^2 + \| \tilde{\sigma} \|_{\infty,k+1}^4) \\
&+ \| F \|_{2,k+1}^2 \, (\| \tilde{\sigma} \|_{\infty,k+1}^4 + \| \tilde{\sigma} \|_{\infty,k+1}^6 \\
&+ \| \tilde{\sigma} \|_{\infty,k+1}^2) + \| F \|_{2,k+2}^2 \, (\| \tilde{\sigma} \|_{\infty,k+1}^2 + \| \tilde{\sigma} \|_{\infty,k+1}^4) \\
&\exp C_7 (\| \dot{\tilde{\sigma}} \|_{2,k+1}^2 \, (1 + \| \tilde{\sigma} \|_{\infty,k+1}^2 + \| \tilde{\sigma} \|_{\infty,k+1}^4) \\
&+ T(\| \tilde{\sigma} \|_{\infty,k+1}^2 + \| \tilde{\sigma} \|_{\infty,k+2}^2 + \| \tilde{\sigma} \|_{\infty,k+1}^4 \\
&+ \| \tilde{\sigma} \|_{\infty,k+1}^6 + \| \tilde{\sigma} \|_{\infty,k+1}^2 \| \tilde{\sigma} \|_{\infty,k+2}^2 \\
&+ \| \tilde{\sigma} \|_{\infty,k+1}^8 \, p\tilde{\sigma} \|_{\infty,k+1}^4 \| \tilde{\sigma} \|_{\infty,k+2}^2))]. \qquad (3.13)
\end{aligned}
$$

Now adding (3.4) and (3.9) multiplied by $2C_5$ and 1 respectively, so that $\| D^{k+2} \nabla v \|_1$ in the right hand side of (3.9) is eliminated, one gets

$$
\begin{aligned}
&\frac{d}{dt} \int_\Omega \tilde{\rho} |D^{k+2} v|^2 + \| D^{k+2} \dot{v} \|_0^2 + \| D^{k+2} \Delta v \|_0^2 \leq \\
&C_9 (\| \dot{\tilde{\sigma}} \|_k^2 \| v \|_{k+2}^2 + \| \tilde{\sigma} \|_{k+1}^4 \| \dot{v} \|_k^2 + \| F \|_{k+2}^2 \\
&+ \| \tilde{\sigma} \|_{k+2}^2 \| \dot{v} \|_k^2 + \| \tilde{\sigma} \|_{k+1}^2 \| F \|_{k+1}^2). \qquad (3.14)
\end{aligned}
$$

Using Gronwall's Lemma, one obtains

$$
\begin{aligned}
&\| v \|_{\infty,k+2}^2 \leq C_{10} (\| v_0 \|_{k+2}^2 + T \| \dot{v} \|_{\infty,k}^2 \, (\| \tilde{\sigma} \|_{\infty,k+2}^4 + \| \tilde{\sigma} \|_{\infty,k+2}^2) \\
&+ \| \tilde{\sigma} \|_{\infty,k+1}^2 \| F \|_{2,k+1}^2 + \| F \|_{2,k+2}^2) \exp C_9 T \| \dot{\tilde{\sigma}} \|_{\infty,k}^2 \, . \qquad (3.15)
\end{aligned}
$$

We integrate (3.4) on $(0, t)$, we obtain

$$
\begin{aligned}
&\| \dot{v} \|_{2,k+2}^2 + \| v \|_{2,k+4}^2 \leq C_3 (\| \tilde{\sigma} \|_{\infty,k+1}^2 \| F \|_{2,k+1}^2 \\
&+ \| F \|_{2,k+2}^2 + T \| \dot{v} \|_{\infty,k}^2 \, (\| \tilde{\sigma} \|_{\infty,k+2}^4 + \| \tilde{\sigma} \|_{\infty,k+2}^2)). \qquad (3.16)
\end{aligned}
$$

Finally by adding (3.13), (3.15) and (3.16), and taking into account (3.13) to substitute $\| \dot{v} \|_{\infty,k}^2$ in the right hand side of (3.15), one obtains (3.3).□
Now we take the following linearized problem in $\sigma$

$$
\left\{
\begin{array}{ll}
\dot{\sigma} - \lambda \Delta \sigma = \tilde{v} . \nabla \tilde{\sigma} & \text{in } Q_T, \\
\nabla \sigma . n = 0 & \text{on } \Sigma_T, \\
\sigma|_{t=0} = \sigma_0(x) & \text{on } \Omega.
\end{array}
\right. \qquad (3.17)
$$

We set $A = \tilde{v} . \nabla \tilde{\sigma}$. We apply the operator $D^k$ to problem (3.17), we obtain

$$
\left\{
\begin{array}{ll}
D^k \dot{\sigma} - \lambda D^k \Delta \sigma = D^k A & \text{in } Q_T, \\
\nabla \sigma . n = 0 & \text{on } \Sigma_T, \\
D^k \sigma|_{t=0} = D^k \sigma_0(x) & \text{on } \Omega.
\end{array}
\right. \qquad (3.18)
$$

In order to get a sufficiently strong estimate for the linearized equation $(3.17)_1$, and as for the estimate (3.3), we will introduce a balance parameter $\epsilon > 0$.

**Theorem 3.5** *Let* $\tilde v \in C((0,T); H^{k+2}(\Omega)) \cap L^2((0,T); H^{k+2}(\Omega))$, $\dot{\tilde v} \in C((0,T); H^k(\Omega)) \cap L^2((0,T); H^{k+2}(\Omega))$, $\tilde\sigma \in C((0,T); H^{k+2}(\Omega)) \cap L^2((0,T); H^{k+2}(\Omega))$, $\dot{\tilde\sigma} \in C((0,T); H^k(\Omega)) \cap L^2((0,T); H^{k+2}(\Omega))$ *and* $\sigma_0(x) \in H^{k+2}_N(\Omega)$. *Then there exists a (unique) strong solution* $\sigma$ *of the problem (3.17). Moreover*

$$\sigma \in C((0,T); H^{k+2}_N(\Omega)) \cap L^2((0,T); H^{k+4}_N(\Omega)),$$

$$\dot\sigma \in L^\infty((0,T); H^k(\Omega)) \cap L^2((0,T); H^{k+2}(\Omega)),$$

*and there exist positive constants* $C_{11}, C_{12}$ *such that the following estimate holds*

$$\begin{aligned}
&\| D^k\dot\sigma \|^2_{\infty,0} + \| D^{k+2}\sigma \|^2_{\infty,0} + \| D^{k+2}\dot\sigma \|^2_{2,0} + \| D^{k+2}\Delta\sigma \|^2_{2,0} \le \\
&C_{11}(\| \sigma_0 \|^2_{k+2} + T \| \tilde v \|^2_{\infty,k+2} (\| \tilde\sigma \|^2_{\infty,k+2} + \| \tilde\sigma \|^4_{\infty,k+2}) \\
&+[1+T(\| \tilde\sigma \|^2_{\infty,k+2} + \| \tilde\sigma \|^4_{\infty,k+1})][\| \dot\sigma_0 \|^2_k + \| \tilde v \|^2_{\infty,k} \| \dot{\tilde\sigma} \|^2_{2,k+1} \\
&+T(\| \dot{\tilde v} \|^2_{\infty,k} \| \tilde\sigma \|^2_{\infty,k+2} + \| \dot{\tilde v} \|^2_{\infty,k+2} (\| \tilde\sigma \|^2_{\infty,k+2} + \| \dot{\tilde\sigma} \|^2_{\infty,k} \\
&+ \| \tilde\sigma \|^4_{\infty,k+1}))] \exp C_{12}T(\| \tilde\sigma \|^2_{\infty,k+2} + \| \tilde\sigma \|^4_{\infty,k+1})).
\end{aligned} \tag{3.19}$$

Before proving this theorem, we will establish some lemmas.

**Lemma 3.6** *One has*

$$\begin{aligned}
&\| D^{k+2}\dot\sigma \|^2_0 + \| D^{k+2}\Delta\sigma \|^2_0 \le C_{13}(\| \dot\sigma \|^2_k (\| \tilde\sigma \|^2_{k+2} + \| \tilde\sigma \|^4_{k+1}) \\
&+ \| \tilde\sigma \|^2_{k+1} \| A \|^2_{k+1} + \| A \|^2_{k+2}).
\end{aligned} \tag{3.20}$$

**Proof**: We make the inner product of $(3.17)_1$ with $(D^k(\tilde\rho\dot\sigma) - \epsilon D^k\Delta\sigma)$. We integrate on $\Omega$ and we obtain

$$\int_\Omega D^k\dot\sigma . D^k(\tilde\rho\dot\sigma) + \lambda\epsilon \| D^k\Delta\sigma \|^2_0 = +\epsilon \int_\Omega D^k\dot\sigma . D^k\Delta\sigma$$

$$+\lambda \int_\Omega D^k(\tilde\rho\dot\sigma) . D^k\Delta\sigma + \int_\Omega D^k A . (D^k(\tilde\rho\dot\sigma) - \epsilon D^k\Delta\sigma).$$

Using

$$D^k(\tilde{\rho}\dot{\sigma}) = D^k\tilde{\rho}\dot{\sigma} + \tilde{\rho}D^k\dot{\sigma} + D^{k-1}(\tilde{\rho}\dot{\sigma}),$$

and

$$m \leq \tilde{\rho} \leq M,$$

one has the following estimate

$$\begin{aligned}
&m \parallel D^k\dot{\sigma} \parallel_0^2 + \lambda\epsilon \parallel D^k\Delta\sigma \parallel_0^2 \leq \\
&C(\epsilon \parallel D^k\dot{\sigma} \parallel_0 \parallel D^k\Delta\sigma \parallel_0 + \lambda \parallel D^k\Delta\sigma \parallel_0 \parallel D^k(\tilde{\rho}\dot{\sigma}) \parallel_0 \\
&+ \parallel D^kA \parallel_0 \parallel D^k(\tilde{\rho}\dot{\sigma}) \parallel_0 + \epsilon \parallel D^kA \parallel_0 \parallel D^k\Delta\sigma \parallel_0 \\
&+(\parallel D^k\tilde{\rho}\dot{\sigma} \parallel_0 + \parallel D^{k-1}(\tilde{\rho}\dot{\sigma}) \parallel_0) \parallel D^k\dot{\sigma} \parallel_0).
\end{aligned} \tag{3.21}$$

Besides, one has

$$\epsilon \parallel D^k\dot{\sigma} \parallel_0 \parallel D^k\Delta\sigma \parallel_0 \leq \frac{\epsilon\lambda}{4} \parallel D^k\Delta\sigma \parallel_0^2 + \frac{\epsilon}{\lambda} \parallel D^k\dot{\sigma} \parallel_0^2,$$

and (see the Appendix lemma 6.2)

$$\parallel D^k(\tilde{\rho}\dot{\sigma} \parallel_0^2 \leq C(\parallel \nabla\tilde{\rho} \parallel_\infty^2 \parallel \dot{\sigma} \parallel_{k-1}^2 + \parallel \nabla\tilde{\rho} \parallel_{k-1}^2 \parallel \dot{\sigma} \parallel_\infty^2).$$

Since $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, one gets

$$\left(\frac{m}{2} - \frac{\epsilon}{\lambda}\right) \parallel D^k\dot{\sigma} \parallel_0^2 + \frac{\lambda\epsilon}{4} \parallel D^k\Delta\sigma \parallel_0^2 \leq$$

$$c(\parallel \tilde{\sigma} \parallel_k^2 \parallel \dot{\sigma} \parallel_2^2 + \parallel A \parallel_k^2 + \parallel \tilde{\sigma} \parallel_3^2 \parallel \dot{\sigma} \parallel_{k-1}^2).$$

By choosing $\epsilon$ so that $\left(\frac{m}{2} - \frac{\epsilon}{\lambda}\right) > 0$, one obtains

$$\begin{aligned}
\parallel D^k\dot{\sigma} \parallel_0^2 + \parallel D^k\Delta\sigma \parallel_0^2 \;\leq\; & C_{13}(\parallel \tilde{\sigma} \parallel_k^2 \parallel \dot{\sigma} \parallel_2^2 \\
& + \parallel A \parallel_k^2 + \parallel \tilde{\sigma} \parallel_3^2 \parallel \dot{\sigma} \parallel_{k-1}^2).
\end{aligned} \tag{3.22}$$

We rewrite (3.22) at the order $(k+1)$ and $(k+2)$, one has

$$\begin{aligned}
\parallel D^{k+1}\dot{\sigma} \parallel_0^2 + \parallel D^{k+1}\Delta\sigma \parallel_0^2 \;\leq\; & C_{13}(\parallel \tilde{\sigma} \parallel_{k+1}^2 \parallel \dot{\sigma} \parallel_2^2 \\
& + \parallel A \parallel_{k+1}^2 + \parallel \tilde{\sigma} \parallel_3^2 \parallel \dot{\sigma} \parallel_k^2),
\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}\parallel D^{k+2}\dot\sigma \parallel_0^2 \quad + \quad & \parallel D^{k+2}\Delta\sigma \parallel_0^2 \leq C_{13}(\parallel \tilde\sigma \parallel_{k+2}^2 \parallel \dot\sigma \parallel_2^2 \\ & + \parallel A \parallel_{k+2}^2 + \parallel \tilde\sigma \parallel_3^2 \parallel \dot\sigma \parallel_{k+1}^2). \qquad (3.24)\end{aligned}$$

Finally, using that $k \geq 2$ and taking into account (3.23) to substitute $\parallel \dot\sigma \parallel_{k+1}^2$ in the right hand side of (3.24), one obtains (3.20).□

**Lemma 3.7** *One has*

$$\frac{d}{dt} \parallel D^k\dot\sigma \parallel_0^2 + \lambda \parallel D^k\nabla\dot\sigma \parallel_0^2 \leq C_{14}(\parallel D^{k+2}\dot\sigma \parallel_0^2 + \parallel \dot A \parallel_k^2). \qquad (3.25)$$

**Proof**: We take the time derivative of $(3.18)_1$, we make the inner product with $D^k\dot\sigma$, we integrate on $\Omega$ and we obtain

$$\int_\Omega D^k\frac{d}{dt}\dot\sigma.D^k\dot\sigma - \lambda\int_\Omega D^k\Delta\dot\sigma.D^k\dot\sigma = \int_\Omega D^k\dot A.D^k\dot\sigma.$$

By integrating by parts the second term on the left hand side, one has

$$\int_\Omega D^k\Delta\dot\sigma.D^k\dot\sigma + \int_\Omega |D^k\nabla\dot\sigma|^2 = < \gamma_n(D^k\nabla\dot\sigma), \gamma_0 D^k\dot\sigma >_{\frac{-1}{2},\frac{1}{2}} . \qquad (3.26)$$

Estimating the other terms and using the equivalence norms, one obtains (3.25).□

**Lemma 3.8** *One has*

$$\frac{d}{dt} \parallel D^{k+2}\sigma \parallel_0^2 + \lambda \parallel D^{k+2}\nabla\sigma \parallel_0^2 \leq C_{15}(\parallel D^{k+2}\nabla\sigma \parallel_1^2 + \parallel A \parallel_{k+2}^2). \qquad (3.27)$$

**Proof**: We make the inner product of $(3.17)_1$ with $D^k\sigma$ and we integrate on $\Omega$. Making the same calculations as before, we obtain (3.27).□
**Proof of Theorem 3.5**: Now, we are able to proof the Theorem 3.5. By adding (3.20) and (3.25) multiplied respectively by $2C_{14}$ and 1, such that $\parallel D^{k+2}\dot\sigma \parallel_0^2$ will be eliminated in the right hand side of (3.25), then using Gronwall's Lemma we obtain

$$\begin{aligned}& \parallel D^k\dot\sigma \parallel_{\infty,0}^2 \leq C_{16}(\parallel \dot\sigma_0 \parallel_k^2 + \parallel \dot A \parallel_{2,k}^2 \, T(\parallel A \parallel_{\infty,k+2}^2 \\ & + \parallel \tilde\sigma \parallel_{\infty.k+1}^2 \parallel A \parallel_{\infty,k+1}^2)) \exp C_{17}T(\parallel \tilde\sigma \parallel_{\infty,k+2}^2 + \parallel \tilde\sigma \parallel_{\infty,k+1}^4). (3.28)\end{aligned}$$

Adding (3.20) and (3.27) multiplied respectively by $2C_{15}$ and 1, such that $\| D^{k+4}\sigma \|_0^2$ will be eliminated in the right hand side of (3.27), then integrating on $(0,T)$ one obtains

$$\begin{aligned}\| D^{k+2}\tilde{\sigma} \|_{\infty,0}^2 \leq C_{18}(\| \sigma_0 \|_{k+2}^2 + T(\| \dot{\sigma} \|_{\infty,k}^2 (\| \tilde{\sigma} \|_{\infty,k+2}^2 \\ + \| \tilde{\sigma} \|_{\infty,k+1}^4) + \| A \|_{\infty,k+2}^2 + \| \tilde{\sigma} \|_{\infty,k+1}^2 \| A \|_{\infty,k+1}^2)). \qquad (3.29)\end{aligned}$$

Now, integrating (3.20) on $(0,T)$ one gets

$$\begin{aligned}\| D^{k+2}\dot{\sigma} \|_{2,0}^2 + \| D^{k+2}\Delta\sigma \|_{2,0}^2 \leq C_{19}T(\| \tilde{\sigma} \|_{\infty,k+1}^2 \| A \|_{\infty,k+1}^2 \\ + \| A \|_{\infty,k+2}^2 + \| \dot{\sigma} \|_{\infty,k}^2 (\| \tilde{\sigma} \|_{\infty,k+2}^2 + \| \tilde{\sigma} \|_{\infty,k+1}^4)). \qquad (3.30)\end{aligned}$$

On the other hand, one has that $A = \tilde{v}.\nabla\tilde{\sigma}$ and $\dot{A} = \dot{\tilde{v}}.\nabla\tilde{\sigma} + \tilde{v}.\nabla\dot{\tilde{\sigma}}$, hence by using the Lemma 6.2 (in the appendix) and $H^2 \hookrightarrow L^\infty$, one gets

$$\begin{aligned}\| D^k\dot{A} \|_0^2 &\leq C_{20}(\| \dot{\tilde{v}} \|_{k-1}^2 \| \tilde{\sigma} \|_4^2 + \| \dot{\tilde{v}} \|_2^2 \| \tilde{\sigma} \|_{k+2}^2 \\ &\quad + \| \tilde{v} \|_3^2 \| \dot{\tilde{\sigma}} \|_k^2 + \| \tilde{v} \|_k^2 \| \dot{\tilde{\sigma}} \|_3^2), \qquad (3.31)\end{aligned}$$

and

$$\| D^k A \|_0^2 \leq C_{21}(\| \tilde{v} \|_3^2 \| \tilde{\sigma} \|_k^2 + \| \tilde{v} \|_k^2 \| \tilde{\sigma} \|_3^2). \qquad (3.32)$$

Finally, adding (3.28), (3.29) and (3.30) and taking into account the estimates (3.31) and (3.32), one obtains (3.19).□

# 4 The nonlinear problem. Local existence.

In this section we solve (1.2) by proving the existence of a fixed point $(\rho, v) = (\tilde{\rho}, \tilde{v})$ for system (3.2), (3.17).
Take $0 < T < \infty$ and define

$$\begin{aligned}R_T = \{&(\tilde{\rho},\tilde{v}),\ \tilde{v} \in L^\infty(0,T;H^{k+2}(\Omega)) \cap L^2(0,T;H^{k+4}(\Omega)), \\ &\dot{\tilde{v}} \in L^\infty(0,T;H^k(\Omega)) \cap L^2(0,T;H^{k+2}(\Omega)), \\ &\tilde{\sigma} \in L^\infty(0,T;H^{k+2}(\Omega)) \cap L^2(0,T;H^{k+4}(\Omega)), \\ &\dot{\tilde{\sigma}} \in L^\infty(0,T;H^k(\Omega)) \cap L^2(0,T;H^{k+2}(\Omega)), \\ &\| \dot{\tilde{v}} \|_{\infty,k}^2 + \| \tilde{v} \|_{\infty,k+2}^2 + \| \dot{\tilde{v}} \|_{2,k+2}^2 + \| \tilde{v} \|_{2,k+4}^2 \leq B, \\ &\tilde{v}|_{t=0} = v_0(x),\ \tilde{v}|_\Gamma = 0,\ \mathrm{div}\tilde{v} = 0 \text{ on } Q_T, \\ &\| \dot{\tilde{\sigma}} \|_{\infty,k}^2 + \| \tilde{\sigma} \|_{\infty,k+2}^2 + \| \dot{\tilde{\sigma}} \|_{2,k+2}^2 + \| \tilde{\sigma} \|_{2,k+4}^2 \leq B, \\ &\tilde{\sigma}|_{t=0} = \sigma_0(x),\ \| \tilde{\sigma} - \sigma_0 \|_\infty \leq \frac{m}{2},\ \nabla\tilde{\sigma}.n|_\Gamma = 0\}. \qquad (4.1)\end{aligned}$$

Consider now the map $\varphi$ defined in $R_T$ in this way,

$$\varphi : (\tilde{v}, \tilde{\rho}) \longrightarrow (v, \rho),$$

where $v$ and $\rho$ are the solution of (3.2) and (3.17) respectively with

$$\begin{aligned} F(\tilde{\rho}, \tilde{v}) &= -\tilde{\rho}(\tilde{v}.\nabla)\tilde{v} + \lambda[(\tilde{v}.\nabla)\nabla\tilde{\rho} + (\nabla\tilde{\rho}.\nabla)\tilde{v}] \\ &\quad + \frac{\lambda^2}{\tilde{\rho}}[(\nabla\tilde{\rho}.\nabla)\nabla\tilde{\rho} - \frac{1}{\tilde{\rho}}(\nabla\tilde{\rho}.\nabla\tilde{\rho})\nabla\tilde{\rho} + \Delta\tilde{\rho}\nabla\tilde{\rho}] \\ &\quad - p_\rho\nabla\tilde{\rho} + \tilde{\rho}f, \end{aligned}$$

$$\tilde{\rho} = \tilde{\sigma} + \overline{\rho}, \quad \sigma_0 = \rho_0 - \overline{\rho}, \quad \text{and} \quad p_\rho = \frac{dp}{d\rho}.$$

We want to prove that $\varphi$ has a fixed point in $R_T$. This point will clearly be a solution of problem (1.2). We evaluate the $H^{k+2}(\Omega)$ norm of $F(\tilde{\rho}, \tilde{v})$ and the $H^k(\Omega)$ norm of $\dot{F}(\tilde{\rho}, \tilde{v})$.

$$\begin{aligned} \| F \|_{k+2}^2 \leq\ & B_1(\| \tilde{\sigma} \|_3^2 \| \tilde{v} \|_{k+1}^2 \| \tilde{v} \|_{k+2}^2 + \| \tilde{\sigma} \|_{k+2}^2 \| \tilde{v} \|_2^2 \| \tilde{v} \|_3^2 \\ & + \| \tilde{\sigma} \|_{k+3}^2 \| \tilde{v} \|_3^2 + \| \tilde{\sigma} \|_4^2 \| \tilde{v} \|_{k+2}^2 \\ & + \| \tilde{\sigma} \|_3^2 \| \tilde{\sigma} \|_{k+2}^2 \| \tilde{\sigma} \|_{k+3}^2 + \| \tilde{\sigma} \|_3^2 \| \tilde{\sigma} \|_4^2 \| \tilde{\sigma} \|_{k+2}^2 \\ & + \| \tilde{\sigma} \|_{k+2}^2 \| \tilde{\sigma} \|_{k+3}^2 + \| \tilde{\sigma} \|_3^2 \| \tilde{\sigma} \|_4^2 + \| \tilde{\sigma} \|_{k+2}^6 \\ & + \| \tilde{\sigma} \|_3^2 \| \tilde{\sigma} \|_{k+2}^6 + \| \tilde{\sigma} \|_3^6 \| \tilde{\sigma} \|_{k+2}^2 + \| \tilde{\sigma} \|_3^6 \\ & + \| \tilde{\sigma} \|_3^4 + \| \tilde{\sigma} \|_{k+2}^4 + \| f \|_{k+1}^4 + \| p_\rho \|_{C^{k+2}}^2 \| \tilde{\sigma} \|_{k+3}^2), \end{aligned}$$

$$\begin{aligned} \| \dot{F} \|_{k-1}^2 \leq\ & B_2(\| \dot{\tilde{\sigma}} \|_2^2 \| \tilde{v} \|_k^4 + \| \dot{\tilde{\sigma}} \|_{k-2}^2 \| \tilde{v} \|_3^2 \| \tilde{v} \|_4^2 \\ & + \| \tilde{\sigma} \|_3^2 \| \dot{\tilde{v}} \|_{k-2}^2 \| \tilde{v} \|_{k+1}^2 + \| \tilde{\sigma} \|_{k-1}^2 \| \dot{\tilde{v}} \|_2^2 \| \tilde{v} \|_3^2 \\ & + \| \tilde{\sigma} \|_k^2 \| \dot{\tilde{v}} \|_k^2 \| \tilde{v} \|_2^2 + \| \dot{\tilde{v}} \|_k^2 \| \tilde{v} \|_k^2 + \| \dot{\tilde{v}} \|_k^2 \| \tilde{\sigma} \|_{k+1}^2 \\ & + \| \tilde{\sigma} \|_{k+1}^2 \| \tilde{v} \|_k^2 + \| \dot{\tilde{\sigma}} \|_k^2 \| \tilde{v} \|_{k+1}^2 + \| \dot{\tilde{\sigma}} \|_k^2 \| \tilde{\sigma} \|_{k+2}^2 \| \tilde{\sigma} \|_k^2 \\ & + \| \dot{\tilde{\sigma}} \|_k^2 \| \tilde{\sigma} \|_{k+2}^2 + \| \dot{\tilde{\sigma}} \|_{k+1}^2 \| \tilde{\sigma} \|_{k+1}^2 \| \tilde{\sigma} \|_k^2 \\ & + \| \dot{\tilde{\sigma}} \|_{k+1}^2 \| \tilde{\sigma} \|_{k+1}^2 + \| \dot{\tilde{\sigma}} \|_k^2 \| \tilde{\sigma} \|_{k+1}^4 \| \tilde{\sigma} \|_k^2 \\ & + \| \dot{\tilde{\sigma}} \|_k^2 \| \tilde{\sigma} \|_{k+1}^4 + \| \dot{\tilde{\sigma}} \|_k^4 + \| \tilde{\sigma} \|_k^4 + \| f \|_{k-1}^4 \\ & + \| \dot{f} \|_{k-1}^2 + \| \dot{f} \|_{k-1}^4 + \| p_\rho \|_k^2 \| \dot{\tilde{\sigma}} \|_k^2). \end{aligned}$$

Consequently one has

$$\| F \|^2_{2.k+2} \le h_1(B)(\sqrt{T}(1+ \| p_\rho \|^2_{C^{k+2}}) + T + \| f \|^4_{4,k+1}), \qquad (4.2)$$

and

$$\| F \|^2_{2.k+1} \le h_2(B)(T(1+ \| p_\rho \|^2_{C^{k+1}}) + \| f \|^4_{4,k}),$$

$$\begin{aligned} \| \dot{F} \|^2_{2.k-1} \quad \le \quad & h_3(B)(T(1+ \| p_\rho \|^2_{C^k}) + \sqrt{T} + \| f \|^4_{4,k-1} \\ & + \| \dot{f} \|^4_{4,k-1} + \| \dot{f} \|^2_{2,k-1}). \end{aligned}$$

In addition

$$\| \dot{\tilde{\sigma}} \|^2_{2,k+1} \le c\sqrt{T} \| \dot{\tilde{\sigma}} \|_{\infty,k} \| \dot{\tilde{\sigma}} \|_{2,k+2},$$

$$\| \dot{v}_0 \|^2_k \le \frac{1}{m}\left(\mu \| v_0 \|^2_{k+2} + \| F(0) \|^2_k\right),$$

and

$$\begin{aligned} \| F(0) \|^2_k \quad \le \quad & (\| v_0 \|^2_{k+1} \| \sigma_0 \|^2_{k+1} + \| \sigma_0 \|^2_{k+2} \| v_0 \|^2_k + \| \sigma_0 \|^4_{k+2} \\ & + \| \sigma_0 \|^4_{k+2} \| \sigma_0 \|^2_{k+1} + \| p_\rho \|^2_{C^k} \| \sigma_0 \|^2_k + \| f(0) \|^2_k). \end{aligned}$$

Hence, if we take

$$\begin{aligned} B \ge \max \quad \{ \quad & 2C_{11}(2 \| \sigma_0 \|^2_{k+2} + \| v_0 \nabla \sigma_0 \|^2_k), 2C_0((1+\mu/m) \| v_0 \|^2_{k+2} \\ & + \| \sigma_0 \|^2_{k+1} \| \sigma_0 \|^2_{k+1} + \| \sigma_0 \|^2_{k+2} \| v_0 \|^2_k + \| \sigma_0 \|^4_{k+2} \\ & + \| \sigma_0 \|^4_{k+2} \| \sigma_0 \|^2_{k+1} + \| p_\rho \|^2_{C^k} \| \sigma_0 \|^2_k + \| f(0) \|^2_k)\}, \quad (4.3) \end{aligned}$$

and $T$ small enough, we get

$$\| \dot{v} \|^2_{\infty,k} + \| v \|^2_{\infty,k+2} + \| \dot{v} \|^2_{\infty,k+2} + \| v \|^2_{2,k+4} \le B, \qquad (4.4)$$

$$\| \dot{\sigma} \|^2_{\infty,k} + \| \sigma \|^2_{\infty,k+2} + \| \dot{\sigma} \|^2_{\infty,k+2} + \| \sigma \|^2_{2,k+4} \le B. \qquad (4.5)$$

The estimate for the sup norm of $(\sigma - \sigma_0)$ in $Q_T$ is proved in [1]. By choosing a small value for $T$ one gets $\| \sigma - \sigma_0 \|_\infty \le \frac{m}{2}$.

Now we utilize Schauder's fixed point theorem. Clearly $R_T$ is a convex, compact set in $X = L^2(0,T;L^2(\Omega)) \times L^2(0,T;L^2(\Omega))$. Since $\varphi(R_T) \subset R_T$, it is sufficient to prove that $\varphi : R_T \to R_T$ is continuous in $X$. Suppose that

$$(\tilde{v}_n, \tilde{\sigma}_n) \in R_T, \quad (\tilde{v}_n, \tilde{\sigma}_n) \longrightarrow (\tilde{v}, \tilde{\sigma}) \ \text{ in } X,$$

and set

$$(\tilde{v},\tilde{\sigma}) = \varphi(\tilde{v}_n,\tilde{\sigma}_n), \quad (v,\sigma) = \varphi(\tilde{v},\tilde{\sigma}).$$

Take the difference between the equations for $(v_n,\sigma_n)$ and $(v,\sigma)$, multiply by $(v_n - v)$ and $(\sigma_n - \sigma)$ respectively, and integrate in $\Omega$. By an energy argument we prove that $(v_n,\sigma_n)$ converges to $(v,\sigma)$ in $X$. Hence $\varphi$ is continuous and it has a fixed point which is the solution of problem (1.2) in $Q_T$.

# 5 Global existence. Asymptotic behavior.

Now we return to the nonlinear problem, with $F = F(\rho, v)$ and $A = (v.\nabla)\sigma$. We add (3.9), (3.10), (3.25) and (3.27) with some appropriate constants so that $\| D^{k+2}\nabla v \|_1^2$, $\| D^k\nabla \dot{v} \|_1^2$, $\| \dot{\sigma} \|_k^2$, $\| D^{k+2}\dot{\sigma} \|_0^2$, and $\| D^{k+4}\sigma \|_0^2$ are eliminated in the right hand side of the inequality.
If we set

$$\Phi(t) = \| \sqrt{\rho} D^k \dot{v} \|_0^2 + \| \sqrt{\rho} D^{k+2} v \|_0^2 + \| D^k \dot{\sigma} \|_0^2 + \| D^{k+2}\sigma \|_0^2, \tag{5.1}$$

and

$$\psi(t) = \| \dot{v} \|_{k+2}^2 + \| v \|_{k+4}^2 + \| \dot{\sigma} \|_{k+2}^2 + \| \sigma \|_{k+4}^2, \tag{5.2}$$

we obtain the following estimate

$$\begin{aligned} \frac{d}{dt}\Phi(t) + \psi(t) &\leq K_1 \Big(\psi(t)(\Phi(t) + \Phi^2(t) + \Phi^6(t)\Big) \\ &+ K_1 \Big(\| f \|_{k+1}^4 + \| \dot{f} \|_{k-1}^4 + \| \dot{f} \|_{k-1}^2\Big). \end{aligned} \tag{5.3}$$

Obviously one has

$$\psi(t) \geq K_2 \Phi(t), \quad \text{with} \quad K_2 \leq 1. \tag{5.4}$$

**Lemma 5.1** *Let $\Gamma \in C^{k+2}$, $(v,\sigma)$ be the solution of (1.1) in $Q_T$. Suppose that*

$$\Phi(0) \leq \frac{\gamma}{K_1}, \qquad \gamma \in ]0, 1/2], \tag{5.5}$$

$$\| f \|_{\infty,k+1}^2 + \| \dot{f} \|_{\infty,k-1}^2 \leq \frac{15K_2}{64K_1^2}\gamma. \tag{5.6}$$

*Then for all t in [0,T]*

$$\Phi(t) \leq \frac{\gamma}{K_1}. \tag{5.7}$$

**Proof**: By the absurd.□
On the other hand, from Sobolev embedding theorem $H^{k+2}(\Omega) \hookrightarrow C(\overline{\Omega})$, one sees that there exists a constant $K_3$ small enought such that,

$$\Phi(t) \leq K_3,$$

then

$$\frac{m}{2} \leq \sigma(t,x) + \overline{\rho} \leq \frac{3}{2}m \quad \in \overline{\Omega}. \tag{5.8}$$

We can prove now the existence of a global solution of (1.2), under the condition that the initial data and the external force are small enough. We can apply again the theorem (local existence), and we find a solution in $[T^*, 2T^*]$, since $v(T^*, x)$ and $\sigma(T^*, x)$ satisfy the same estimates than $v_0(x)$ and $\sigma_0(x)$. We can repeat this argument in each interval $[0, nT^*]$, $n \in N$ and consequently we obtain the existence of a global solution.□
Now we suppose that $f = 0$, then we obtain from (5.3)

$$\begin{aligned} \frac{d}{dt}\Phi(t) &\leq -\psi\left[1 - K_1(\Phi(t) + \Phi^2(t) + \Phi^6(t))\right], \\ &\leq \psi K_1\left(\Phi(t) + \Phi^2(t) + \Phi^6(t)\right) - \psi, \end{aligned}$$

According to (5.7) and the fact that $K_1 \geq 1$ and $\gamma \leq \frac{1}{2}$, one has

$$K_1\left(\Phi(t) + \Phi^2(t) + \Phi^6(t)\right) \leq \frac{15}{64}.$$

On the other hand, by (5.4) one gets the following inequality

$$\forall t \in R^+, \qquad \frac{d}{dt}\Phi(t) \leq -\frac{15K_2}{64}\Phi(t), \tag{5.9}$$

this implies

$$\Phi(t) \leq \Phi(0)\mathrm{e}^{-\frac{15}{64}K_2 t} \quad \forall t \in R^+. \tag{5.10}$$

Using the equivalence norms and $m \leq \rho(t,x) \leq M$, we obtain (2.3).□
**Proof of theorem 2.2** We will use the following result proved by Beirão da Veiga (see [2], Theorem 1.4).

**Theorem 5.2** *Under the assumptions (2.4)-(2.5), and by eventually choosing a smaller constant $k_4$, one has*

$$\| \sigma(t) \|_2^2 \leq k_7 \mathrm{e}^{-k_8 t} \| \sigma_0(t) \|_2^2 \, . \tag{5.11}$$

*If moreover, $(\bar{v}, \bar{\sigma})$ is another solution of problem (1.2) with initial data $(\bar{v}_0, \bar{\sigma}_0)$ satisfying (2.4) then*

$$\| v(t) - \bar{v}(t) \|_s \leq k_9 \mathrm{e}^{-k_{10} t}, \quad \forall t \geq 0, \tag{5.12}$$

*for each fixed $s \in [0, 1[$.*

By an interpolation result one has that

$$\| \sigma(t) \|_r \leq C \| \sigma(t) \|_{k+2}^{1-\theta} \| \sigma(t) \|_2^{\theta} \, . \tag{5.13}$$

Using (5.13) and (5.7), one easily gets (2.6).
Now, by the same interpolation result we have

$$\| v(t) - \bar{v}(t) \|_{r'} \leq C \| v(t) - \bar{v}(t) \|_{k+2}^{1-\theta} \| v(t) - \bar{v}(t) \|_s^{\theta} \, . \tag{5.14}$$

Using (5.14) and (5.7), one obtains (2.7). In order to prove (2.9), under the assumptions of theorem 2.2, it is sufficient to note that the solution of problem (2.8) are just particular solutions of (2.1) corresponding to the case in which the initial density $\rho_0(x)$ is constant (equal to $\overline{\rho}$).□

# 6 Appendix.

For the reader's convenience, we state here some useful results. Here $\Omega$ is the n-dimensional torus, an open bounded regular subset of $R^n$, $R^n$ itself, or $R^n_+ = \{x = (x', x_n), \quad x_n > 0\}$.

**Lemma 6.1** *Let be $r_2 > n/2$, $0 \leq l \leq r_2$, $l \leq l_i \leq r_2 \quad for \quad i = 1..m$, and $l_1 + ... + l_m = l + (m-1) r_2$. Then*

$$\| f_1 ... f_m \|_l \leq C \| f_1 \|_{l_1} \cdots \| f_m \|_{l_m} \, . \tag{6.1}$$

**Lemma 6.2** *Let be $|\alpha| \leq l$. Then*

$$\| D^\alpha (fg) \|_0 \leq C \left( |Df|_\infty \| g \|_{l-1} + |g|_\infty \| Df \|_{l-1} \right) . \tag{6.2}$$

**Lemma 6.3** *Let* $\psi \in C^{r_2}(R;R)$, $r_2 \geq 1$. *Then, there are increasing funtions* $\beta_1 \in C^\infty(R_0^+;R^+)$ *and* $\beta_2 \in C^\infty(R_0^+ \times R_0^+;R^+)$ *such that*

$$\| D^\alpha \psi(g) \|_0^2 \leq \beta_1(|g|_\infty) \| g \|_{r_2}^2, \tag{6.3}$$

*and*

$$\| D^\alpha \psi(g) - D^\alpha \psi(f) \|_0^2 \leq \beta_2(|g|_\infty, |f|_\infty) \| g \|_{r_2}^2, \tag{6.4}$$

*for each* $\alpha, \quad 1 \leq |\alpha| \leq r_2$.

**Proof**: For the proof of the previous lemmas see for instance [2] (the appendix)

**Theorem 6.4** *Let be* $\Omega \in C^{0,1}$ *bounded and let*

$$W = \left\{ v \in L^2(\Omega)^n, \ \mathrm{div} v \in L^2(\Omega) \right\}.$$

*Then*
*(i)* W *is a Hilbert space,*
*(ii)* $D(\Omega)^n$ *is dense in* W,
*(iii) the application* $\gamma_n$ *such that*

$$\begin{array}{rll} \gamma_n : & \mathrm{W} & \longrightarrow H^{-\frac{1}{2}}(\Gamma) \\ & v & \rightarrow v.n = \gamma_n v \end{array}$$

*is continuous. Moreover* $\forall u \in H^1(\Omega)$,

$$\int_\Omega v.\nabla u + \int_\Omega u \mathrm{div} v = < \gamma_n v, \gamma_0 u >_{-\frac{1}{2},\frac{1}{2}} . \tag{6.5}$$

$\gamma_0$ *being the trace operator that maps* $H^1(\Omega)$ *into* $L^2(\Gamma)$.

**Proof**: see [7] (Theorem 1.2, page 9).

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