Upper Bounds on the Rate of Convergence of Truncated Stochastic Infinite-Dimensional Differential Systems with H-Regular Noise

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Abstract

The rate of H-convergence of truncations of stochastic infinite-dimensional systems

$$du = [Au + B(u)]dt + G(u)dW, u(0, \cdot) = u_0 \in H$$

with nonrandom, local Lipschitz-continuous operators A, B and G acting on a separable Hilbert space H, where $u = u(t, x) : [0, T] \times \mathbb{D} \to \mathbb{R}^d$ ($\mathbb{D} \subset \mathbb{R}^d$) is studied. For this purpose, some new kind of monotonicity conditions on those operators and an existing H-series expansion of the space-time noise W are exploited. The rate of convergence is expressed in terms of the converging series-remainder $h(N) = \sum_{k=N+1}^{+\infty} \alpha_n^2$ belonging to the trace of related covariance operator Q of W with eigenvalues $\alpha_n \in \mathbb{R}^1$ of Q. An application to the approximation of semilinear stochastic partial differential equations with cubic-type of nonlinearity is given too.

Key words: Stochastic Partial Differential Equations; Truncated Stochastic Infinite-Dimensional Systems; Space-Time Noise; Stochastic-Numerical Methods; Rate of Convergence; Approximation by Truncation; Eigenfunction-Approach.

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1 Introduction

Consider Itô-type infinite-dimensional stochastic differential systems of the form

$$\begin{aligned} du &= [Au + B(u)]dt + G(u)dW \\ u(0,x) &= u_0(x) \in H, u = u(t,x), t \geq 0, x \in \mathbb{D} \subset \mathbb{R}^d \end{aligned} \tag{1}$$

where $\mu(\mathbb{D}) < +\infty$ of given nonrandom, open, connected subset $\mathbb{D} \subset \mathbb{R}^d$. Suppose that A, B, G are appropriate (pseudo-)differential operators acting on a separable Hilbert space H with scalar product $< .,. >_H$ (and set of scalars from $\mathbb{R}^1$) and satisfying the hypotheses (H0) – (H6) from below among possibly further conditions such that a unique strong solution $u = u(t,x)$ of (1) with $\mathbb{E}\,\|u(t,\cdot)\|_H^2 < +\infty$ exists on $[0,T] \times \mathbb{D}$ where $\mathbb{D} \subseteq \mathbb{R}^d$. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ be a filtered complete probability basis and W an H-valued Wiener process in time with some conditions as given in detail below. The driving space-time noise W is supposed to be H-regular such that

$$W = W(t)(x) = \sum_{n=1}^{+\infty} \alpha_n \beta_n(t) e_n(x) \tag{2}$$

with $\sum_{n=1}^{+\infty} \alpha_n^2 < +\infty$, where $\alpha_n \in \mathbb{R}^1$ are the eigenvalues of related trace class covariance operator Q, β_n are standard independent Wiener processes and $\{e_n : n \in \mathbb{N}\}$ forms an orthonormal system of the separable Hilbert space H. Furthermore, $\mathcal{B}(S)$ denotes the σ-algebra of Borel sets of inscribed set S and μ is the Lebesgue-measure.

Now, consider truncations of (1). For this purpose, suppose that u possesses a H-converging series representation, and let $H^N = span\{e_1, ..., e_N\}$ such that

$$u(t,x) = \sum_{n=1}^{+\infty} c_n(t) e_n(x) \tag{3}$$

can be truncated by

$$u^N(t,x) = \sum_{n=1}^{N} c_n^N(t) e_n(x) \tag{4}$$

where $c_n^N =< u^N, e_n >_H$. Also the H-valued space-time noise W is substituted by

$$W^N = W^N(t)(x) = \sum_{n=1}^{N} \alpha_n \beta_n(t) e_n(x)$$

where β_n are independent standard real-valued Wiener processes. Suppose that the nonlinearity part B needs to be truncated by B^N as well. So the truncation of system (1) is governed by

$$\begin{aligned} du^N &= [Au^N + B^N(u^N)]dt + G(u^N)dW^N \\ u^N(0,x) &= u_0^N(x) \in H \end{aligned} \tag{5}$$

where W^N is the natural finite series truncation of W governed by (2).

2 Main assumptions and an auxiliary result

Suppose that

(H0) $u_0 = u(0,x), u_0^N = u^N(0,x) \in H$ are $\mathcal{F}_0$-measurable, $\mathbb{E}\,[\|u_0\|_H^2 + \|u_0^N\|_H^2] < +\infty$, and the initial well-posedness $u_0, u_0^N \in D(A) \cap D(B) \cap D(B^N) \cap D(G)$ where $D(\cdot)$ denotes the domain of definition of inscribed operators.

(H1) $W(t)$ as the space-time Wiener process takes its values in its own separable Hilbert space $U \subset H$ and it has a trace class covariance operator $Q : U \longmapsto U$ with $\mathbb{E}\,[W(s) \otimes W(t)] = (t \wedge s)Q$, where $Q : U \longmapsto U$ is a positive definite, self-adjoint, bounded operator with finite trace. Moreover, if we denote by α_k the eigenvalues of Q then the trace of Q is $\sum_{n=1}^{+\infty} \alpha_n^2$ and we may write

$$W(t) = \sum_{n=1}^{+\infty} \alpha_n \beta_n e_n$$

as an expansion of $W(t)$ where the β_n's are independent standard real valued Wiener process. The convergence is guaranteed with respect to the naturally induced metric on U. Furthermore, there is a function $h : \mathbb{N} \to \mathbb{R}_+^1$ satisfying $\lim_{N\to+\infty} h(N) = 0$ and $2\sum_{n=N+1}^{+\infty} \alpha_n^2 \leq h(N)$.

(H2) The linear (unbounded) operator $A : D(A) \subset H \longmapsto H$ is self-adjoint and generates a strongly continuous semigroup $\{S(t) : t \geq 0\}$ of bounded operators on H such that there are nonrandom, nonnegative coefficient functions $c_{bA}, c_{mA}, \varepsilon_{A,1}, \varepsilon_{A,2} \in L^1([0,T], \mathcal{B}([0,T]), \mu)$ satisfying

$$\begin{aligned} < A(u), u >_H &\leq -c_{bA}(t)\|u\|_H^2 - \varepsilon_{A,1}(t)\|A^{1/2}(u)\|_H^2, \\ < A(u-v), u-v >_H &\leq -c_{mA}(t)\|u-v\|_H^2 - \varepsilon_{A,2}(t)\|A^{1/2}(u-v)\|_H^2 \end{aligned}$$

for all $u, v \in D(A)$. (Note that one may take $c_{bA} = c_{mA}$ and $\varepsilon_{A,1} = \varepsilon_{A,2}$ due to the imposed linearity of A.)

(H3) The local Lipschitz-continuous nonlinear operator $B : D(B) \subseteq H \longmapsto H$ together with its Lipschitz-continuous truncation $B^N : D(B^N) \subseteq H^N \longmapsto H^N$ are well-defined and possess nonrandom coefficient functions $c_0^2, c_{bB}, c_{mB}, \varepsilon_{B,1}, \varepsilon_{B,2} \in L^1([0,T], \mathcal{B}([0,T]), \mu)$ satisfying

$$\begin{aligned}
<B(u), u>_H &\leq c_0^2(t) + c_{bB}(t)\|u\|_H^2 + \varepsilon_{B,1}(t)\|A^{1/2}(u)\|_H^2,\\
<B^N(u), u>_H &\leq c_0^2(t) + c_{bB}(t)\|u\|_H^2 + \varepsilon_{B,1}(t)\|A^{1/2}(u)\|_H^2,\\
<B(u)-B(v), u-v>_H &\leq c_{mB}(t)\|u-v\|_H^2 + \varepsilon_{B,2}(t)\|A^{1/2}(u-v)\|_H^2,\\
<B^N(u)-B^N(v), u-v>_H &\leq c_{mB}(t)\|u-v\|_H^2 + \varepsilon_{B,2}(t)\|A^{1/2}(u-v)\|_H^2,\\
<B(u)-B^N(v), u-v>_H &\leq c_{mB}(t)\|u-v\|_H^2 + \varepsilon_{B,2}(t)\|A^{1/2}(u-v)\|_H^2
\end{aligned}$$

for all $u, v \in D(B) \cap D(B^N)$ (usually B^N such that $D(B) \subseteq D(B^N)$).

(H4) The local Lipschitz continuous, linearly bounded functional $G : D(G) \subseteq H \longmapsto \mathbb{R}^1$ possesses nonnegative, nonrandom coefficient functions $c_{bG}, c_{mG}, \varepsilon_{G,1}, \varepsilon_{G,2} \in L^1([0,T], \mathcal{B}([0,T]), \mu)$ satisfying

$$\begin{aligned}
\|G(u)\|_H^2 &\leq c_{bG}(t)(1 + \|u\|_H^2) + \varepsilon_{G,1}(t)\|A^{1/2}(u)\|_H^2,\\
\|G(u) - G(v)\|_H^2 &\leq c_{mG}(t)\|u-v\|_H^2 + \varepsilon_{G,2}(t)\|A^{1/2}(u-v)\|_H^2
\end{aligned}$$

for all $u, v \in D(G)$.

(H5) $\forall n \geq N_0 \geq 1 \, \forall t \in [0,T]$ we have

$$2\varepsilon_{A,2}(t) - 2\varepsilon_{B,2}(t) - \varepsilon_{G,1}(t) \sum_{n=N+1}^{+\infty} \alpha_n^2 - \varepsilon_{G,2}(t) \sum_{n=1}^{N} \alpha_n^2 \geq 0 \quad \text{and}$$

$$2\varepsilon_{A,1}(t) - 2\varepsilon_{B,1}(t) - \varepsilon_{G,1}(t) \sum_{n=1}^{+\infty} \alpha_n^2 \geq 0.$$

(H6) Initial regularity of the approximation problem holds, i.e.

$$\mathbb{E}\,\|u_0 - u_0^N\|_H^2 \leq \frac{1}{2} h(N)(1 + \max_{0 \leq t \leq T} \mathbb{E}\,\|u(t,\cdot)\|_H^2).$$

It may be noted that conditions (H0) – (H5) also guarantee the existence of a global unique strong solution $u \in H$ of system (1) and its truncation (5) with finite second moments $\mathbb{E}\,\|u(t,\cdot)\|_H^2 < +\infty$. For details, see [7,8], [10], [24], [27], [31] and [33].

Theorem 2.1 *We assume that hypotheses (H0) - (H5) are satisfied. Then, for any strong solution u of (1) or (5), we have*

$$\forall 0 \leq t \leq T \ : \ \mathbb{E}\,\|u(t,\cdot)\|_H^2 \leq \Big(\mathbb{E}\,\|u(0,\cdot)\|_H^2 + K_0(T)\Big)\exp(K_1(T)) \tag{6}$$

where $K_0 \geq 0, K_1 : [0,T] \to \mathbb{R}^1$ *are in* $L^1([0,T], \mathcal{B}([0,T]), \mu)$ *and satisfy*

$$K_0(T) \leq \int_0^T \left[2c_0^2(s) + \sum_{n=1}^{+\infty} \alpha_n^2 c_{bG}(s)\right] ds \ \ \textit{and}$$

$$K_1(T) \leq 2 \int_0^T \Big([-c_{bA}(s) + c_{bB}(s) + \frac{1}{2} \sum_{n=1}^{+\infty} \alpha_n^2 c_{bG}(s)]_+ \Big) ds$$

(nondecreasing in T) where $[z]_+$ denotes the positive part of inscribed expression z.

PROOF. Suppose that (H0) is satisfied. Apply Itô formula to system (1) (See [15]). This implies that

$$d\|u\|_H^2 = 2(<Au, u>_H + <B(u), u>_H)dt + 2 <G(u)dW, u>_H + \sum_{n=1}^{+\infty} \alpha_n^2 \|G(u)\|_H^2 dt.$$

It is not difficult to see that $\int_0^t < G(u)dW, u >_H$ forms a square-integrable martingale with zero expectation. Taking expectation under hypothesis (H0) - (H4) leads to the differential inequality

$$\begin{aligned} d\,\mathbb{E}\,\|u\|_H^2 \leq & - \left[2\Big(\varepsilon_{A,1}(t) - \varepsilon_{B,1}(t)\Big) - \varepsilon_{G,1}(t) \sum_{n=1}^{+\infty} \alpha_n^2 \right] \mathbb{E}\,\|A^{1/2}u\|_H^2 dt + \\ & +2 \left[-c_{bA}(t) + c_{bB}(t) + \frac{1}{2} \sum_{n=1}^{+\infty} \alpha_n^2 c_{bG}(t) \right] \mathbb{E}\,\|u\|_H^2 dt + \\ & + \left[2c_0^2(t) + \sum_{n=1}^{+\infty} \alpha_n^2 c_{bG}(t) \right] dt. \end{aligned}$$

Suppose that (H5) is satisfied. Then, the maximum solution $v(t) \geq \mathbb{E}\,\|u(t,\cdot)\|_H^2$ of above differential inequality is bounded by

$$v(t) \leq \left(v(0) + 2 \int_0^T c_0^2(s) ds + \sum_{n=1}^{+\infty} \alpha_n^2 \int_0^T c_{bG}(s) ds \right) \exp\Big(K_2(T)\Big)$$

where

$$K_2(T) \leq 2 \int_0^T \Big([-c_{bA}(s) + c_{bB}(s) + \frac{1}{2} \sum_{n=1}^{+\infty} \alpha_n^2 c_{bG}(s)]_+ \Big) ds.$$

Consequently, the assertion of Theorem 2.1 is confirmed. ◇

3 General theorem on rate of strong H-convergence

This section establishes our main result under the hypotheses (H0) – (H6) and conditions that guarantee the existence of strong solutions of (1).

Theorem 3.1 *We assume that the hypotheses (H0) - (H6) are satisfied. Then, for any strong solution u of (1), we have*

$$\forall N \geq N_0 \, \forall 0 \leq t \leq T \;:\; \mathbb{E}\, \|u(t,\cdot) - u^N(t,\cdot)\|_H^2 \leq h(N) C_0(T) \exp(C_1(T)) \tag{7}$$

where $\lim_{N\to+\infty} h(N) = 0$, *and the constants* $C_i(T)$ *can be estimated uniformly by*

$$C_0(T) \leq \Big(1 + \max_{0\leq t\leq T} \mathbb{E}\, \|u(t,\cdot)\|_H^2\Big) \cdot \Big(1 + \int_0^T c_{bG}(t)dt\Big),$$

$$C_1(T) \leq 2 \int_0^T \Big([-c_{mA}(s) + c_{mB}(s) + \frac{1}{2}\sum_{n=1}^{+\infty} \alpha_n^2 c_{mG}(s)]_+\Big) ds$$

(nondecreasing in T) where $[z]_+$ denotes the positive part of inscribed real number z.

PROOF. Subtract equation (5) from (1) to obtain the differential equation

$$\begin{aligned} d(u-u^N) = &\Big[A(u-u^N) + B(u) - B^N(u^N)\Big]dt + \\ &+ G(u)d(W-W^N) + (G(u)-G(u^N))dW^N. \end{aligned} \tag{8}$$

Recall that $W - W^N = \sum_{n=N+1}^{+\infty} \alpha_n \beta_n e_n$ and $W^N = \sum_{n=1}^{N} \alpha_n \beta_n e_n$ are independent and orthogonal on $(H, <\cdot,\cdot>_H)$. Now, apply Itô-formula to (8) in order to find the differential of $\|u - u^N\|_H^2$. This yields that

$$\begin{aligned} d\|u-u^N\|_H^2 = &\Big[2 < A(u-u^N), u-u^N >_H + 2 < B(u) - B^N(u^N), u-u^N >_H\Big]\, dt + \\ &+ 2 < G(u)d(W-W^N), u-u^N >_H + 2 < (G(u)-G(u^N))dW^N, u-u^N >_H + \\ &+ \|G(u)\|_H^2 \sum_{n=N+1}^{+\infty} \alpha_n^2 dt + \|G(u) - G(u^N)\|_H^2 \sum_{n=1}^{N} \alpha_n^2 dt. \end{aligned} \tag{9}$$

Obviously, $< G(u)d(W-W^N), u-u^N >_H + < (G(u)-G(u^N))dW^N, u-u^N >_H$ are square-integrable martingales with vanishing expectation. Now, take the expectation at both sides of (10) and apply the monotonicity conditions (H2) - (H4) to get to the estimates

$$
\begin{aligned}
& d\,\mathbb{E}\,\|u-u^N\|_H^2 \qquad\qquad (10)\\
& \leq -\left[\Big(2\varepsilon_{A,2}(t)-2\varepsilon_{B,2}(t)-\varepsilon_{G,1}(t)\sum_{n=N+1}^{+\infty}\alpha_n^2-\varepsilon_{G,2}(t)\sum_{n=1}^{N}\alpha_n^2\Big)\,\mathbb{E}\,\|A^{1/2}(u-u^N)\|_H^2\right]dt+\\
& \quad +\left[2\Big(-c_{mA}(t)+c_{mB}(t)+\frac{1}{2}\sum_{n=1}^{N}\alpha_n^2 c_{mG}(t)\Big)\,\mathbb{E}\|u-u^N\|_H^2+\sum_{n=N+1}^{+\infty}\alpha_n^2 c_{bG}(t)(1+\mathbb{E}\|u\|_H^2)\right]dt.
\end{aligned}
$$

Recall hypothesis (H5) for $n \geq N_0$. This leads to the differential inequality

$$
dv_N(t) \leq \begin{cases} 2\Big(-c_{mA}(t)+c_{mB}(t)+\dfrac{1}{2}\displaystyle\sum_{n=1}^{N}\alpha_n^2 c_{mG}(t)\Big)v_N(t)dt \\ +\displaystyle\sum_{n=N+1}^{+\infty}\alpha_n^2 c_{bG}(t)\Big(1+\max_{0\leq t\leq T}\,\mathbb{E}\,\|u(t,\cdot)\|_H^2\Big)dt \end{cases}
$$

for $v_N(t) = \mathbb{E}\,\|u(t,\cdot)-u^N(t,\cdot)\|_H^2$. Its maximum solution is bounded since, by Gronwall-Bellman-technique, we obtain

$$
v_N(t) \leq \left(v_N(0)+\sum_{n=N+1}^{+\infty}\alpha_n^2\Big(1+\max_{0\leq s\leq t}\,\mathbb{E}\,\|u(s,\cdot)\|_H^2\Big)\int_0^t c_{bG}(s)ds\right)\cdot\exp\Big(C_1(t)\Big)
$$

where

$$
C_1(t) \leq 2\int_0^t\Big([-c_{mA}(s)+c_{mB}(s)+\frac{1}{2}\sum_{n=1}^{N}\alpha_n^2 c_{mG}(s)]_+\Big)ds.
$$

Suppose that $v_N(0) \leq h(N)(1+\max_{0\leq t\leq T}\,\mathbb{E}\,\|u(t,\cdot)\|_H^2)/2$ as required by (H6). Therefore, the truncation error $v(t)$ satisfies

$$
v_N(t) \leq h(N)C_0(t)\exp\Big(2\int_0^t([-c_{mA}(s)+c_{mB}(s)+\frac{1}{2}\sum_{n=1}^{N}\alpha_n^2 c_{mG}(s)]_+)ds\Big)
$$

where

$$
C_0(t) \leq \Big(1+\max_{0\leq s\leq t}\,\mathbb{E}\,\|u(s,\cdot)\|_H^2\Big)\Big(1+\int_0^t c_{bG}(s)ds\Big).
$$

Hence, the assertion of Theorem 3.1 is proven. ◇

Remark 3.2 *Theorem 3.1 is important in order to control the total error of numerical approximations of SPDEs since it says how to truncate the infinite-series solutions of SPDEs by finite-series solutions of SPDEs with finite-dimensional noise. The remaining problem is to find the coefficients $c_n^N(t)$ in truncation u^N defined by (4). Often the latter task leads to the approximation of inifinite systems of ordinary SDEs whose L^2-error can be controlled by general theorems from [30]. For more details on numerical approximations of finite-dimensional systems of SDE, see [29].*

Remark 3.3 *For numerical approximations, by Theorem 3.1, we obtain the control on the rate of convergence by taking* $h(N) = (1/N)^{\gamma}$ *or* $h(N) = (1/ln(N))^{\gamma}$ *with rate* $\gamma > 0$ *in the above assumptions. A similar estimate of the space-error for numerical approximations of parabolic SPDEs could be found in [18] (p. 36). There the approximation error in space was found to be controlled by the leading remainder eigenvalue* λ_{N+1} *of operator* A*, hence this case corresponds to the special case* $h(N) = |\lambda_{N+1}|^{-2\gamma}$ *for operators with monotonically decreasing eigenvalues* $\lambda_{N+1} < 0$ *such as* $A = \triangle$*.*

4 Application to approximation of equations with cubic nonlinearity

For modelling nonlinear dynamics in spatio-temporal optical chaos (laserdynamics, nanotechnology), one encounters noisy reaction-diffusion equations of the type

$$\begin{aligned} du &= [a^2\Delta u + u(1 - ||u||_{L^2}^2)]dt + G(u)dW_t \\ u(0) &= u_0 \in H = L_{loc}^2(D) \end{aligned} \tag{11}$$

where $D \subseteq \mathbb{R}^d$ and $a \geq 0$ a certain diffusivity constant. Often the noise intensity $G(u)$ can be modelled by

$$G(u) = \sigma_0 + \sigma_1||u||_{L^2} + \sigma_2||(-\Delta u)^{1/2}||_{L^2} \tag{12}$$

with real constants σ_i. The existence and uniqueness of strong solutions of (11) is clear from [31]. Now, one is interested in studying the effect of truncation W^N of the space-time noise W and truncated initial data u_0^N on the error $u - u^N$. For this purpose, one can easily apply our main result of Theorem 3.1. For its application, one needs to check its assumptions (H0) - (H6). Doing this, we may take

$$Au = a^2\Delta u, \quad B(u) = u(1 - ||u||_H^2), \quad B^N(u) = B(u)$$

with the domain of definition $D(A) \subseteq H^2(\mathbb{R}^d)$. Let us check the assumptions (H2)-(H4) whereas the assumptions (H0)-(H1) are rather obvious. (H2) is satisfied by the observation

$$< Au, u >_H = -a^2||\nabla u||_H^2$$

where the linearity of A gives the other estimate on continuity. Assumption (H3) is verified by

$$< B(u), u >_H = ||u||_H^2 - ||u||_H^4 \;\leq\; ||u||_H^2$$

and

$$
\begin{aligned}
< B(u)-B(v), u-v >_H &= ||u-v||_H^2 - < ||u||_H^2 u - ||v||_H^2 v, u-v >_H \\
&= ||u-v||_H^2 - 3\eta^2 ||u-v||_H^2 \ \leq \ ||u-v||_H^2
\end{aligned}
$$

where η is an intermediate value between

$$\min(||u||_H, ||v||_H) \ \leq \ \eta \leq \max(||u||_H, ||v||_H).$$

It remains to check (H4). We have

$$
\begin{aligned}
||G(u)||_H^2 &\leq 3\sigma_0^2 + 3\sigma_1^2 ||u||_H^2 + 3\sigma_2^2 ||(-\Delta u)^{1/2}||_H^2 \\
&\text{and} \\
||G(u)-G(v)||_H^2 &\leq 2\sigma_1^2 ||u-v||_H^2 + 2\sigma_2^2 ||(-\Delta u)^{1/2} - (\Delta v)^{1/2}||_H^2
\end{aligned}
$$

for all $u, v \in D(G) \subseteq H = L^2_{loc}(\mathbb{R}^d)$ with compact support. Now, we may set

$$\varepsilon_{A,1}(t) = a^2, \quad c_{bA}(t) = 0, \quad c_{mA}(t) = 0, \quad \varepsilon_{A,2}(t) = a^2,$$

$$c_0(t) = 0, \quad c_{bB}(t) = 1, \quad \varepsilon_{B,1}(t) = 0, \quad c_{mB}(t) = 1, \quad \varepsilon_{B,2}(t) = 0,$$

$$c_{bG}(t) = 3\max\{\sigma_0^2, \sigma_1^2\}, \quad \varepsilon_{G,1}(t) = 3\sigma_2^2, \quad c_{mG}(t) = 2\sigma_1^2, \quad \varepsilon_{G,2}(t) = 2\sigma_2^2.$$

Thus, (H5) reads as

$$2a^2 - 3\sigma_2^2 \sum_{n=N+1}^{+\infty} \alpha_n^2 - 2\sigma_2^2 \sum_{n=1}^{N} \alpha_n^2 \geq 0 \quad \text{and} \quad 2a^2 - 3\sigma_2^2 \sum_{n=1}^{+\infty} \alpha_n^2 \ \geq \ 0$$

which exhibits the interplay between diffusivity and noise intensities. Recall that α_n are the eigenvalues belonging to the covariance operator of W. Suppose that (H6) is satisfied. Consequently, all assumptions (H0)-(H6) can be fulfilled. Therefore, we may apply our basic result on the upper estimate of rate of convergence of approximations (e.g. Galerkin or other finite-dimensional approximations) and this rate is controlled by $h(N)$ which is essentially determined by the rate of convergence of the covariance of the driving noise W. Notice that (H5) is trivially satisfied whenever $\sigma_2 = 0$, otherwise extreme care is needed (even the existence of strong solutions is problematic restricted to sufficiently small σ_2 in this case).

Remark 4.1 (On Numerical Methods for SPDEs) *Approximations of SPDEs have been studied from quite different perspectives. [2], [3] and [16] considered splitting techniques, [6] and [32] for parabolic SPDEs, [9] and [23] by Galerkin-type approximations with time-dependent noise, [11], [12] and [13] by lattice approximation, [17], [18] and [19] for stochastic evolution equations, [28] by difference schemes for hyperbolic equations, and [34] in* $\mathbb{R}^1$ *by semi-discretization through difference methods using discrete Sobolev space techniques. To the best of our knowledge, we are not aware of any publication other than ours where the rate of convergence of the* L^2*-error of approximations of general stochastic evolution equations is studied under the above fairly general hypotheses (H0) – (H6). Our main result can also be understood as a* *little contribution to understand better how truncations of noise terms, initial data or nonlinearities occurring in stochastic evolution equations effect the total L^2-error caused by numerical approximation procedures.*

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