Stationary solutions for the 2D stochastic dissipative Euler equation

Hakima Bessaih

Abstract. A 2-dimensional dissipative Euler equation, subject to a random perturbation is considered. Using compactness arguments, existence of martingale stationary solutions are proved.

Mathematics Subject Classification (2000). Primary 60H15, Secondary 76D05.

1. Introduction

We are concerned with the dissipative Euler equations for an incompressible fluid perturbed by a multiplicative noise, in an open bounded domain D of R^2 with a smooth boundary ∂D which satisfies the locally Lipschitz condition see [1], i.e.

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p - \chi u + f + G(u)\zeta, \tag{1.1}$$

where u is the velocity of the fluid, p the pressure, f the external force, ζ is a Gaussian random field white noise in time, subject to the restrictions imposed below and G is an operator acting on solution. The constant χ will be called the sticky viscosity. u is subject to the incompressibility condition

$$\nabla \cdot u(t,x) = 0, \quad t \in [0,T], \quad x \in D, \tag{1.2}$$

the boundary condition

$$u \cdot n = 0 \quad \text{on } \partial D, \tag{1.3}$$

n being the external vector. When $\chi = 0$, (1.1) is the classical Euler equation. For an additive noise, existence of strong solutions (in the probabilistic sense) has been proved in [3] for a bounded domain, in [15] in the whole space and in [8] on the torus. For a multiplicative noise, existence of martingale solutions can be found in [4] and [7].

2. Notations, hypothesis and main result

Let $\mathcal{V}$ be the space of infinitely differentiable vector fields u on D with compact support strictly contained in D, satisfying $\nabla \cdot u = 0$. We introduce the space H of all measurable vector fields $u : D \longrightarrow \mathbb{R}^2$ which are square integrable, divergence free, and tangent to the boundary

$$H = \left\{ u \in \left[L^2(D)\right]^2 ;\ \nabla \cdot u = 0 \text{ in } D,\ u \cdot n = 0 \text{ on } \partial D \right\}.$$

The space H is a separable Hilbert space with the inner product inherited from $\left[L^2(D)\right]^2$, denoted in the sequel by $< .,. >$ (norm $|.|$). Let V be the following subspace of H

$$V = \left\{ u \in \left[H^1(D)\right]^2 ;\ \nabla \cdot u = 0 \text{ in } D,\ u \cdot n = 0 \text{ on } \partial D \right\}.$$

The space V is a separable Hilbert space with the inner product inherited from $\left[H^1(D)\right]^2$ (norm $\| \ . \ \|$). Identifying H with its dual space H', and H' with the corresponding natural subspace of the dual space V', we have the standard triple $V \subset H \subset V'$ with continuous dense injections. We denote the dual pairing between V and V' by the inner product of H.

Let $b(\cdot,\cdot,\cdot) : V \times V \times V \longrightarrow \mathbb{R}$ be the continuous trilinear form defined as

$$b(u, v, z) = \int_D (u \cdot \nabla v) \cdot z.$$

It is well known that there exists a continuous bilinear operator $B(\cdot,\cdot) : V \times V \longrightarrow V'$ such that

$$< B(u, v), z >= b(u, v, z), \text{ for all } z \in V.$$

By the incompressibility condition, we have

$$< B(u, v), v >= 0 \text{ and } \quad < B(u, v), z >= - < B(u, z), v > .$$

Let K be another separable Hilbert space. Denote by $L_2(K, H)$ the set of Hilbert-Schmidt operators from K to H.

Let $p > 1$ and m a nonnegative integer, $W^{m,p}$ are the Sobolev spaces. When $p = 2$ then $W^{m,p}$ will be denoted by H^m. Let $0 < \alpha < 1$ then $W^{\alpha,p}(0, T; H)$ is the Sobolev space of all $u \in L^p(0, T; H)$ such that

$$\int_0^T \int_0^T \frac{|u(t) - u(s)|^p}{|t - s|^{1+\alpha p}} dtds < \infty.$$

We impose throughout the paper the following conditions:

1. $W(t)$ is a K-cylindrical Wiener process.
2. $f \in V$.

let us assume that

$$(\mathbf{G1}) \quad G : V \longrightarrow L_2(K, V), \text{ is globally Lipschitz continuous}$$

$$(\mathbf{G2}) \quad \begin{cases} |G(u)|^2_{L_2(K,H)} \leq \lambda_0 |u|^2 + \rho_0, \\ |\nabla \wedge G(u)|^2_{L_2(K,H)} \leq \lambda_1 |\nabla \wedge u|^2 + \lambda_2 |u|^2 + \rho_1, \ \forall u \in V \end{cases}$$

where $\nabla \wedge u = D_1 u_2 - D_2 u_1$ and $\lambda_0, \lambda_1, \lambda_2, \rho_0, \rho_1$ are positive constants independent of u.

Now let us give the following definition of stationary martingale solution

Definition 2.1. A martingale solution of the equation (1.1) is a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbf{P})$, a K-cylindrical Wiener process W and a progressively measurable process $u : [0,\infty) \times \Omega \to H$, with $\mathbf{P}$-a.e.paths

$$u(.,\omega) \in C([0,T], D(A^{-\alpha/2})) \cap L^\infty(0,T;V)$$

for all $T > 0$, and $\alpha > 1$ such that $\mathbf{P}$-a.s. the identity

$$\begin{aligned} &< u(t), v > + \int_0^t < B(u(s), u(s)), v > ds + \chi \int_0^t < u(s), v > ds \\ &=< u(0), v > + \int_0^t < f(s), v > ds + < \int_0^t G(u(s)) dW(s), v > \end{aligned} \tag{2.1}$$

holds true for all $t \geq 0$ and all $v \in \mathcal{V}$. The space $D(A^{-\alpha/2})$ will be defined in the next section.

Moreover, a stationary martingale solution of equation (1.1) is a martingale solution such that the process is stationary in H.

Remark 2.2. A function belonging to $C([0,T], D(A^{-\alpha/2})) \cap L^\infty(0,T;V)$ is weakly continuous in H. Hence, for every $t \geq 0$, the mapping $\omega \to u(t,\omega)$ is well defined from Ω to H and it is weakly measurable. Since H is a separable Banach space, it is strongly measurable see [18] pp 131. Therefore, it is meaningful to speak about the law of $u(t)$ in H. The stationarity of u in H introduced above has to be understood in this sense.

The existence of martingale solutions has been proved in [4] and in [7]. Here, we are interested in stationary martingale solutions.

Theorem 2.3. *In addition to the assumptions (*$\mathbf{G1}$*) and (*$\mathbf{G2}$*), assume that*

$$\chi > \frac{3}{2}\lambda_0 \quad \text{and} \quad \chi > \frac{\lambda_1}{2}$$

*then (*1.1*) has a stationary martingale solution.*

3. The dissipative Navier-Stokes approximation.

For every $\nu > 0$, we consider the equation of Navier-Stokes type

$$\begin{cases} \frac{\partial u}{\partial t} + (u\cdot\nabla)u + \nabla p = \nu\Delta u - \chi u + f + G(u)\frac{\partial W}{\partial t}, & \text{in } (0,T)\times D \\ \nabla\cdot u = 0, & \text{in } (0,T)\times D \\ \nabla\wedge u = 0, & \text{on } (0,T)\times \partial D \\ u\cdot n = 0, & \text{on } (0,T)\times \partial D \\ u|_{t=0} = u_0, & \text{in } D \end{cases} \tag{3.1}$$

Let $a(\cdot,\cdot) : V\times V \longrightarrow \mathbb{R}$ be the bilinear continuous form defined in [2] as

$$a(u,v) = \int_D \nabla u\cdot\nabla v - \int_{\partial D} k(\sigma)u(\sigma)\cdot v(\sigma)d\sigma,$$

where $k(\sigma)$ is a function defined on the boundary ∂D, and we have the following estimates, see [13] for the details

$$\int_{\partial D} k(\sigma)u(\sigma)\cdot v(\sigma)d\sigma \le C \parallel u \parallel\parallel v \parallel,$$

and for an arbitrary $\epsilon > 0$

$$\int_{\partial D} k(\sigma)|u(\sigma)|^2 d\sigma \le \epsilon \parallel u \parallel^2 + C(\epsilon)|u|^2. \tag{3.2}$$

Moreover, we set

$$D(A) = \left\{u\in V\cap (H^2(D))^2, \nabla\wedge u = 0\right\},$$

and define the linear operator $A : D(A) \longrightarrow H$, as

$$Au = -\Delta u.$$

We will denote the domain of A^α by $D(A^\alpha)$. Here $D(A^{-\alpha/2})$ denotes the dual of $D(A^{\alpha/2})$, and we perform identification as above to have

$$D(A^{\alpha/2}) \subset V \subset H \subset V' \subset D(A^{-\alpha/2}).$$

In place of equations (3.1) we will consider the abstract stochastic evolution equation

$$\begin{cases} du(t) + \nu Au(t)dt + B(u(t),u(t))dt = -\chi u(t)dt + f(t)dt + G(u(t))dW(t) \\ u(0) = u_0, \end{cases} \tag{3.3}$$

for $t\in[0,T]$. Assume that (**G1**) and (**G2**) hold and let $\alpha > 1$ be fixed. We have the following continuous embedding, see [1]-pp 85, thm 4.12 part II,

$$D(A^{\alpha/2}) \subset [H^\alpha(D)]^2 \subset [C(\bar{D})]^2.$$

Let P_n be the operator from $D(A^{-\alpha/2})$ to $D(A^{\alpha/2})$ defined as

$$P_n x = \sum_{i=1}^n < x, e_i > e_i, \quad x\in D(A^{-\alpha/2})$$

Let $B_n(u,u)$ be the Lipschitz operator in P_nH defined as

$$B_n(u,u) = \pi_n B(u,u), \quad u \in P_nH,$$

where $\pi_n : H \to [0,1]$ is a C^∞ function and is defined as $\pi_n(u) = 1,\ |u| \leq n$ and $\pi_n(u) = 0,\ |u| \geq n+1$.

Consider the classical Faedo-Galerkin approximation scheme defined by the processes $u_{n\nu}(t) \in P_nH$, solutions of

$$\begin{cases} du_{n\nu}(t) + \nu A u_{n\nu}(t)dt + P_n B_n(u_{n\nu}(t), u_{n\nu}(t))dt = -\chi u_{n\nu} \\ \qquad\qquad P_n f(t)dt + P_n G(u_{n\nu}(t))dW(t), \\ \\ u_{n\nu}(0) = P_n u_0. \end{cases} \tag{3.4}$$

$t \in [0,T]$.

Lemma 3.1. *There exist positive constants $C_1(p)$ and $\tilde{C}_1$ independent on n and on ν such that for each $p \geq 2$*

$$\mathbf{E}(\sup_{0\leq s\leq t} |u_{n\nu}(s)|^p) \leq C_1(p), \tag{3.5}$$

and moreover

$$\nu \int_0^t \mathbf{E}\|u_{n\nu}(s)\|^2)ds \leq \tilde{C}_1. \tag{3.6}$$

Proof. By Itô formula, for $p \geq 2$ we have

$$\begin{aligned} d|u_{n\nu}(t)|^p &\leq p|u_{n\nu}(t)|^{p-2} < u_{n\nu}, du_{n\nu} > \\ &+ \frac{1}{2}p(p-1)|u_{n\nu}(t)|^{p-2}|G(u_{n\nu})|^2_{L_2(K,V)}dt. \end{aligned}$$

Since $< B(u_{n\nu}, u_{n\nu}), u_{n\nu} >= 0$ and using the hypothesis (**G2**) we get

$$\begin{aligned} d|u_{n\nu}(t)|^p &+ \nu p|u_{n\nu}(t)|^{p-2}|\nabla u_{n\nu}|^2 + \chi p|u_{n\nu}(t)|^p \leq \\ &\nu p|u_{n\nu}(t)|^{p-2}\int_{\partial D} k|u_{n\nu}|^2 dt + p|u_{n\nu}(t)|^{p-2} < f, u_{n\nu} > dt \\ &+ (1/2)p(p-1)|u_{n\nu}(t)|^{p-2}(\lambda_0|u_{n\nu}(t)|^2 + \rho_0)dt \\ &+ p|u_{n\nu}(t)|^{p-2} < G(u_{n\nu})dW, u_{n\nu} > . \end{aligned} \tag{3.7}$$

Using the Hölder inequality and then the Young inequality for the second term on the right hand side of the above inequality, for a fixed $\epsilon_1 > 0$ we obtain

$$\begin{aligned} |u_{n\nu}(t)|^{p-2} < f, u_{n\nu} > &\leq |u_{n\nu}(t)|^{p-1}|f| \\ &\leq \epsilon_1|u_{n\nu}(t)|^p + C(\epsilon_1,p)|f|^p, \end{aligned}$$

Using Young inequality for the third term, for a fixed $\epsilon_2 > 0$ we get

$$\frac{1}{2}p(p-1)|u_{n\nu}(t)|^{p-2}\rho_0 \leq \epsilon_2|u_{n\nu}(t)|^p + C(\epsilon_2,p).$$

Thus, by using (3.2) and the previous estimates, we have

$$
\begin{aligned}
& d|u_{n\nu}(t)|^p + \nu p(1-\epsilon)|u_{n\nu}(t)|^{p-2}|\nabla u_{n\nu}|^2 dt + \chi p|u_{n\nu}|^p dt \leq \\
& C(\epsilon_1,p)|f|^p dt + C(\epsilon_2,p)dt + p|u_{n\nu}(t)|^{p-2} < G(u_{n\nu})dW, u_{n\nu} > \\
+ \quad & \left(\frac{\lambda_0}{2}p(p-1) + \epsilon_2 + \epsilon_1 + \nu p C_\epsilon\right)|u_{n\nu}(t)|^p dt
\end{aligned}
$$

Now we integrate over $(0,t)$, take the supremum on t and integrate over Ω, we obtain

$$
\begin{aligned}
\mathbf{E}(\sup_{0\leq s\leq t}|u_{n\nu}(s)|^p) \quad \leq \quad & \mathbf{E}(|u_{n\nu}(0)|^p) \\
+ \quad & \left(\frac{\lambda_0}{2}p(p-1) + \epsilon_2 + \epsilon_1 + \nu p C_\epsilon - p\chi\right)\int_0^t \mathbf{E}(\sup_{0\leq s\leq r}|u_{n\nu}(s)|^p)dr \\
+ \quad & C(\epsilon_2,p)t + C(\epsilon_1,p)\int_0^t \mathbf{E}|f|^p ds \\
+ \quad & p\mathbf{E}\left(\sup_{0\leq s\leq t}\int_0^s |u_{n\nu}(r)|^{p-2} < G(u_{n\nu})dW(r), u_{n\nu}(r) >\right).
\end{aligned}
$$

Let us estimate the last term in the above inequality. By Burkholder-Davis-Gundy inequality see [9] pp 82 thm 3.14, we get

$$
\begin{aligned}
& p\mathbf{E}\left(\sup_{0\leq s\leq t}\int_0^s |u_{n\nu}(r)|^{p-2} < G(u_{n\nu}(r))dw(r), u_{n\nu}(r) >\right) \leq \\
& p\mathbf{E}\left(\int_0^t |u_{n\nu}(r)|^{2p-2}|G(u_{n\nu}(r))|^2_{L_2(K,V)}dr\right)^{1/2}.
\end{aligned}
$$

Using (**G2**) in the above inequality and the Cauchy-Schwartz's inequality, we get

$$
\begin{aligned}
& p\mathbf{E}\left(\int_0^t |u_{n\nu}(r)|^{2p-2}|G(u_{n\nu}(r))|^2_{L_2(K,V)}dr\right)^{1/2} \\
\leq \quad & p\mathbf{E}\left(\int_0^t (\lambda_0|u_{n\nu}(r)|^{2p} + \rho_0|u_{n\nu}(r)|^{2p-2})dr\right)^{1/2} \\
\leq \quad & p\mathbf{E}\left(\sup_{0\leq s\leq t}|u_{n\nu}(s)|^{p/2}\left(\int_0^t (\lambda_0|u_{n\nu}(r)|^p + \rho_0|u_{n\nu}(r)|^{\frac{2p-2}{p}}dr\right)^{1/2}\right) \\
\leq \quad & \frac{1}{2}\mathbf{E}\left(\sup_{0\leq s\leq t}|u_{n\nu}(s)|^p\right) + \frac{p^2}{2}\mathbf{E}\int_0^t \lambda_0 \sup_{0\leq s\leq \sigma}|u_{n\nu}(s)|^p + \frac{p^2}{2}\rho_0\mathbf{E}\int_0^t |u_{n\nu}(s)|^{\frac{2p-2}{p}}ds.
\end{aligned}
$$

Finally, we estimate the last term in the above inequality using the Young's inequality. For $\epsilon_3 > 0$ we obtain

$$
\frac{p^2}{2}\rho_0\mathbf{E}\int_0^t |u_{n\nu}(s)|^{\frac{2p-2}{p}}ds \leq \epsilon_3\int_0^t |u_{n\nu}(s)|^p ds + C(\epsilon_3,p).
$$

Collecting all the estimates, we obtain that

$$\frac{1}{2}\mathbf{E}(\sup_{0\le s\le t}|u_{n\nu}(s)|^p)\le \mathbf{E}(|u_{n\nu}(0)|^p)+C_2\int_0^t \mathbf{E}(\sup_{0\le s\le r}|u_{n\nu}(s)|^p)dr+C_3. \quad (3.8)$$

where

$$C_2=\frac{\lambda_0}{2}(p(p-1)+p^2)+\epsilon_1+\epsilon_2+\epsilon_3+\nu pC_\epsilon-p\chi,$$

and

$$C_3=C(\epsilon_1,p)\int_0^t \mathbf{E}|f|^p+C(\epsilon_2,p)+C(\epsilon_3,p).$$

Using Gronwall's lemma we get (3.5).

Let us go back to (3.7), take $p=2$ and integrate over $(0,t)$ we get

$$\begin{aligned}2\nu\int_0^t|\nabla u_{n\nu}|^2 &+ 2\chi\int_0^t|u_{n\nu}(t)|^2\le |u_{n\nu}(0)|^2+2\nu\int_0^t\int_{\partial D}k|u_{n\nu}|^2+2<f,u_{n\nu}>dt\\ &+ \int_0^t(\lambda_0|u_{n\nu}(t)|^2+\rho_0)+2\int_0^t<G(u_{n\nu})dW,u_{n\nu}> .\end{aligned}$$

In the above inequality integrate over Ω, then

$$\mathbf{E}\int_0^t<G(u_{n\nu})dW,u_{n\nu}>=0.$$

Now use (3.2) to estimate the second term on the left side and Cauchy-Schwartz inequality to estimate the third term on the left side. Finally, using the estimate (3.5) we get (3.6). □

Lemma 3.2. *There exists a positive constant C_4 which does not depend on n and on ν such that*

$$\mathbf{E}\int_0^t\parallel u_{n\nu}(s)\parallel^2\le C_4, \quad (3.9)$$

Proof. Let $\xi_{n\nu}=\nabla\wedge u_{n\nu}$. We apply the curl to the equation (3.4) we get for $t\in[0,T]$

$$d\xi_{n\nu}+\nu A\xi_{n\nu}dt+\nabla\wedge P_nB_n(u_{n\nu},u_{n\nu})dt=-\chi\xi_{n\nu}dt+\nabla\wedge P_nfdt+\nabla\wedge(G(u_{n\nu}))dW.$$

By Itô formula we have

$$\begin{aligned}d|\xi_{n\nu}|^2 &= 2<\xi_{n\nu},d\xi_{n\nu}>+|\nabla\wedge(G(u_{n\nu}))|^2_{L_2(K,V)}\\ &= -2\nu<A\xi_{n\nu},\xi_{n\nu}>dt-2<\nabla\wedge P_nB_n(u_{n\nu},u_{n\nu}),\xi_{n\nu}>dt\\ &- 2\chi|\xi_{n\nu}|^2+2<\nabla\wedge P_nf,\xi_{n\nu}>dt\\ &+ <\nabla\wedge(G(u_{n\nu}))dw,\xi_{n\nu}>+|\nabla\wedge(G(u_{n\nu}))|^2_{L_2(K,V)}. \quad (3.10)\end{aligned}$$

Since $\xi_{n\nu}|_{\partial D}=0$, $<\nabla\wedge P_nB(u_{n\nu},u_{n\nu}),\xi_{n\nu}>=0$ and using (**G2**), we get that

$$\begin{aligned}d|\xi_{n\nu}|^2 &+ 2\nu|\nabla\xi_{n\nu}|^2dt\le -2\chi|\xi_{n\nu}|^2dt+2<\nabla\wedge P_nf,\xi_{n\nu}>dt\\ &+ <\nabla\wedge(G(u_{n\nu}))dW,\xi_{n\nu}>+\lambda_1|\xi_{n\nu}|^2+\lambda_2|u_{n\nu}|^2+\rho_1.\end{aligned}$$

Now using the Young inequality for the second term on the right hand of the above inequality and for a fixed $\epsilon_4 > 0$ we obtain

$$\begin{aligned} d|\xi_{n\nu}|^2 &+ 2\nu|\nabla\xi_{n\nu}|^2 dt \le (-2\chi + \lambda_1 + \epsilon_4)|\xi_{n\nu}|^2 dt + C(\epsilon_4, p)|\nabla \wedge P_n f| \\ &+ < \nabla \wedge (G(u_{n\nu}))dW, \xi_{n\nu} > + \lambda_2 |u_{n\nu}|^2 + \rho_1. \end{aligned}$$

We integrate over $(0, t)$ and then over Ω. Since

$$\mathbf{E}\int_0^t < \nabla \wedge (G(u_{n\nu}))dW, \xi_{n\nu} >= 0,$$

we obtain the following estimate

$$\begin{aligned} \mathbf{E}|\xi_{n\nu}(t)|^2 &\le \mathbf{E}|\xi_{n\nu}(0)|^2 + (-2\chi + \lambda_1 + \epsilon_4)\mathbf{E}\int_0^t |\xi_{n\nu}(s)|^2 ds \\ &+ C(\epsilon_4)\int_0^t |\nabla \wedge P_n f| + \lambda_2 \int_0^t \mathbf{E}|u_{n\nu}|^2 + \rho_1 t, \end{aligned} \tag{3.11}$$

Using Gronwall Lemma, we obtain that there exists a positive constant C_5 independent of n and of ν such that

$$\mathbf{E}|\xi_{n\nu}(s)|^2 \le C_5. \tag{3.12}$$

Now let us introduce the following elliptic problem

$$\begin{cases} -\Delta u_{n\nu} = \nabla^\perp \xi_{n\nu} & \text{in } D, \\ u_{n\nu} \cdot n = 0 & \text{on } \partial D, \\ \xi_{n\nu} = 0 & \text{on } \partial D, \end{cases} \tag{3.13}$$

where $\nabla^\perp = (D_2, -D_1)$.

We multiply the first equation of (3.13) by $u_{n\nu}$ and integrate over D, we have

$$- < \Delta u_{n\nu}.u_{n\nu} >=< \nabla^\perp \xi_{n\nu}, u_{n\nu} > .$$

By integration by part and in virtue of (3.2), we obtain

$$|\nabla u_{n\nu}(t)|^2 \le \epsilon|\nabla u_{n\nu}(t)|^2 + C_\epsilon |u_{n\nu}(t)|^2 + |\xi_{n\nu}(t)|^2,$$

for all $t \in (0, T)$ and for an arbitrary $\epsilon > 0$. We integrate the above inequality respectively over $(0, t)$ and over Ω, we obtain

$$\begin{aligned} \mathbf{E}\int_0^t |\nabla u_{n\nu}|^2 &\le C\mathbf{E}\left(\int_0^t |u_{n\nu}|^2\right) + \mathbf{E}\left(\int_0^t |\xi_{n\nu}|^2\right) \\ &\le Ct\mathbf{E}\left(\sup_{0\le s\le t} |u_{n\nu}(s)|^2\right) + \mathbf{E}\left(\int_0^t |\xi_{n\nu}|^2\right), \end{aligned}$$

C being a constant independent of n and ν. According to (3.5) and (3.12), this yields the estimate (3.9). □

4. Construction of stationary solutions

Step 1. Take $p = 2$ in(3.8) we get that

$$\begin{aligned}\mathbf{E}|u_{n\nu}(t)|^2 &\leq \mathbf{E}|u_{n\nu}(0)|^2 \\ &+ (\lambda_0 + \epsilon_2 + \epsilon_1 + 2\nu C_\epsilon - 2\chi)\int_0^t \mathbf{E}|u_{n\nu}(s)|^2 ds \\ &+ C(\epsilon_2)t + C(\epsilon_1)\int_0^t \mathbf{E}|f|^2 ds.\end{aligned}$$

If $\chi > \frac{3}{2}\lambda_0$ and $\chi > \frac{\lambda_1}{2}$ then we can choose ϵ_1, ϵ_2, ϵ_4 and ν_0 in the above inequality and in (3.11) such that using Gronwall lemma we get that

$$\mathbf{E}\|u_{n\nu}(t)\|^2 \leq C \quad \forall t \geq 0 \quad \forall n \geq 1 \tag{4.1}$$

for some constant $C > 0$. This implies that there exists an invariant measure for (3.4) by the classical Krylov-Bogoliubov argument (see [10]). Call $\mu_{n\nu}$ one of such invariant measures. From (4.1) we have

$$\int_{P_n V} |x|^2 \mu_{n\nu}(dx) \leq C \quad \forall n \geq 1 \tag{4.2}$$

There exists a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}\}_t, \mathbf{P})$, possibly larger than the one given at the beginning, that supports a random variable $u_{n\nu}(0) - \mathcal{F}_0$ measurable, with law $\mu_{n\nu}$ and a cylindrical Wiener process $W(t)$ with values in K. The solution $\tilde{u}_{n\nu}$ with initial condition $u_{n\nu}(0)$ is a stationary process.

Step 2.

Now let us prove that the family $\{\mathcal{L}(\tilde{u}_{n\nu})\}_{n\nu}$ is tight in $L^2(0,T;H) \cap C([0,T]; D(A^{-\alpha/2}))$, for all given $\alpha > 1$; in fact we decompose $\tilde{u}_{n\nu}$ as

$$\begin{aligned}\tilde{u}_{n\nu}(t) &= \tilde{u}_{n\nu}(0) - \nu\int_0^t A\tilde{u}_{n\nu}(s) - \int_0^t P_n B_n(\tilde{u}_{n\nu}(s), \tilde{u}_{n\nu}(s)) \\ &\quad + \int_0^t P_n f(s) + \int_0^t G(\tilde{u}_{n\nu}(s))dW(s) \\ &= J_1 + ... + J_5.\end{aligned}$$

We have from the bound (4.2) on $\mu_{n\nu}$ that

$$\mathbf{E}|J_1|^2 \leq C_6.$$

From (3.8)

$$\mathbf{E} \parallel J_2 \parallel^2_{W^{1,2}(0,T;V')} \leq C_7.$$

Moreover, we have

$$\mathbf{E} \parallel J_4 \parallel^2_{W^{1,2}(0,T;V')} \leq C_8.$$

for suitable positive constants C_6, C_7, C_8. Using Lemma 5.1 and the uniform assumption (**G1**) and the estimate (3.5) we have

$$\begin{aligned}\mathbf{E} \parallel J_5 \parallel^2_{W^{\gamma,2}(0,T;H)} &\leq \mathbf{E}\int_0^T \parallel G(\tilde{u}_{n\nu}) \parallel^2_{L_2(K,H)} \\ &\leq \mathbf{E}\int_0^T (\lambda_0|\tilde{u}_{n\nu}(s)|^2+\rho_0)ds \\ &\leq C_9(\lambda_0,\rho_0,\gamma)\end{aligned}$$

for $\gamma \in (0,1/2)$, C_9 being independent of n and ν.

Since $\alpha > 1$, $D(A^{\alpha/2}) \subset (L^\infty(D))^2$ so that

$$| < B(u,u), v > | \leq C|u| \parallel u \parallel |A^{\alpha/2}v|,\ \ u \in V,\ \ v \in D(A^{\alpha/2})$$

for some constant $C > 0$. Hence, we have

$$\parallel J_3 \parallel^2_{W^{1,2}(0,T;D(A^{-\alpha/2}))} \leq C_{10} \sup_{0\leq t\leq T} |\tilde{u}_{n\nu}(t)|^2 \int_0^T \parallel \tilde{u}_{n\nu}(s) \parallel^2 ds$$

for some positive constant C_{10} independent of n and ν. In virtue of (3.5) and (3.9), we obtain that

$$\mathbf{E} \parallel J_3 \parallel^2_{W^{1,2}(0,T;D(A^{-\alpha/2}))} \leq C_{11}.$$

Clearly for $\gamma \in (0,1/2)$, $W^{1,2}(0,T;D(A^{-\alpha/2})) \subset W^{\gamma,2}(0,T;D(A^{-\alpha/2}))$, collecting all the previous inequalities we have

$$\mathbf{E} \parallel \tilde{u}_{n\nu} \parallel_{W^{\gamma,2}(0,T;D(A^{-\alpha/2}))} \leq C_{12}, \tag{4.3}$$

for $\gamma \in (0,1/2)$ and $\alpha > 1$, C_{12} being a positive constant independent of n and ν. By (3.9) and (4.3), we have that the laws $\mathcal{L}(\tilde{u}_{n\nu})$ are bounded in probability in

$$L^2(0,T;V) \cap W^{\gamma,2}(0,T;D(A^{-\alpha/2})).$$

Thus by Theorem (5.2), $\{\mathcal{L}(\tilde{u}_{n\nu})\}$ is tight in $L^2(0,T;H)$. On the other hand, by theorem (5.3) $\{\mathcal{L}(\tilde{u}_{n\nu})\}$ is tight in $C([0,T];D(A^{-\beta/2})$, for $\alpha < \beta$.

Step 3. Let us endow $L^2_{loc}(0,\infty;H)$ by the distance

$$d_2(u,v) = \sum_{k=1}^{\infty} 2^{-k} \min\left(|u-v|_{L^2(0,k;H)}, 1\right),$$

and similarly $C(0,\infty;D(A^{-\beta/2}))$ by the distance

$$d_\infty(u,v) = \sum_{k=1}^{\infty} 2^{-k} \min\left(|u-v|_{C[0,k];D(A^{-\beta/2}))}, 1\right).$$

Hence, we obtain that $\{\mathcal{L}(\tilde{u}_{n\nu})\}_{n\nu}$ is tight in $L^2_{loc}(0,\infty;H) \cap C([0,\infty];D(A^{-\beta/2}))$, thus $\tilde{u}_{n\nu}$ is a stationary solution in H. Let us choose $\nu = 1/n$. From Prokhorov Theorem (see [9] p32), the set of the laws $\{\mathcal{L}(\tilde{u}_{n\nu})\}$ is relatively compact. By Skorohod Theorem, there exists a basis $(\Omega^1, \mathcal{F}^1, \{\mathcal{F}^1_t\}_{t\geq 0}, \mathbf{P}^1)$ and on this basis, $L^2_{loc}(0,\infty;H) \cap C([0,\infty];D(A^{-\beta/2}))$-valued random variables u^1, $u^1_{n\nu}$, such that $\mathcal{L}(\tilde{u}_{n\nu}) = \mathcal{L}(u^1_{n\nu})$, on $L^2_{loc}(0,\infty;H) \cap C([0,\infty];D(A^{-\beta/2}))$, and $u^1_{n\nu} \longrightarrow u^1$ $\mathbf{P^1}$-a.s. in $L^2_{loc}(0,\infty;H) \cap C([0,\infty];D(A^{-\beta/2}))$. Since $u^1_{n\nu}$ and $\tilde{u}_{n\nu}$ have the same law, $u^1_{n\nu}$ is also a stationary solution. By the a.s. convergence, u^1 is a stationary solution in H.

By (3.5) and (3.9) we have

$$\mathbf{E}(\sup_{0\le s\le t} |u^1_{n\nu}(s)|^p) \le C_1(p),$$

$$\mathbf{E}(\int_0^t \| u^1_{n\nu}(s) \|^2) \le C_2,$$

for all $n \ge 1$ and $p \ge 2$. Hence, we have that

$$u^1(.,\omega) \in L^2_{loc}(0,\infty;V) \cap L^\infty_{loc}(0,\infty;H)\mathbf{P} - a.s.$$

and $u^1_{n\nu} \longrightarrow u^1$ weakly in $L^2(\Omega\times(0,\infty);V)$. Let us define the process $M_{n\nu}(t)$ with trajectories in $C([0,\infty];H)$ as

$$\begin{aligned} M_{n\nu}(t) = u^1_{n\nu}(t) \quad &- \quad P_n u^1 + \nu \int_0^t A u^1_{n\nu}(s)ds + \int_0^t P_n B_n(u^1_{n\nu}(s), u^1_{n\nu}(s))ds \\ &- \quad \int_0^t P_n f(s)ds. \end{aligned}$$

We will prove that $M_{n\nu}(t)$ is a square integrable martingale with respect to the filtration

$$\sigma\left\{u^1_{n\nu}(s), s \le t\right\},$$

with quadratic variation

$$<< M_{n\nu} >>_t = \int_0^t G(u^1_{n\nu})G(u^1_{n\nu})^* ds. \tag{4.4}$$

We shall prove the following Lemma

Lemma 4.1. *Assume that (3.5) and (3.9) hold then*

$$\left\langle \int_0^t P_n B_n(u^1_{n\nu}(s), u^1_{n\nu}(s))ds, v \right\rangle \longrightarrow \left\langle \int_0^t B(u^1(s), u^1(s))ds, v \right\rangle$$

for all $t \in [0,\infty)$ *and* $v \in \mathcal{V}$ $\mathbf{P}$-*a.s.*

Proof.

$$\begin{aligned} \left\langle \int_0^t P_n B_n(u^1_{n\nu}(s), u^1_{n\nu}(s))ds, v \right\rangle \quad &= \quad \left\langle \int_0^t \pi_n(u^1_{n\nu}(s))(u^1_{n\nu}(s))_i D_i(u^1_{n\nu}(s))_j ds, v_j \right\rangle \\ &= \quad -\int_0^t \int_D \pi_n(u^1_{n\nu}(s))(u^1_{n\nu})_i(s)(u^1_{n\nu})_j \frac{\partial (v_n)_j(s)}{\partial x_i} \end{aligned}$$

That converges $\mathbf{P}$-a.s. to

$$\int_0^t \int_D (u^1)_i(s)(u^1)_j \frac{\partial (v)_j(s)}{\partial x_i} = \left\langle \int_0^t B(u^1(s), u^1(s))ds, v \right\rangle.$$

□

Since $u_{n\nu}$ and $u^1_{n\nu}$ have the same law, for φ be a real valued, bounded and continuous function on $C([0,s];D(A^{-\beta/2}))$ where $0 \le s \le t \le T$ and all $v, z \in \mathcal{V}$, we have

$$\mathbf{E}(< M_{n\nu}(t) - M_{n\nu}(s), v > \varphi(u_{n\nu})) = 0 \tag{4.5}$$

and

$$\begin{aligned}\mathbf{E} \quad &((< M_{n\nu}(t), v >< M_{n\nu}(t), z > - < M_{n\nu}(s), v >< M_{n\nu}(s), z > \\ &- \int_s^t G(u^1_{n\nu}(r))G(u^1_{n\nu}))^* \varphi(u^1_{n\nu})) = 0.\end{aligned} \tag{4.6}$$

By (3.5), (3.9) we can take the limit in (4.5) and (4.6) and we obtain

$$\mathbf{E}(< M^1(t) - M^1(s), v > \varphi(u_{n\nu})) = 0 \tag{4.7}$$

and

$$\begin{aligned}\mathbf{E} \quad &((< M^1(t), v >< M^1(t), z > - < M^1(s), v >< M^1(s), z > \\ &- \int_s^t G(u^1(r))G(u^1(r)^* \varphi(u^1)) = 0,\end{aligned} \tag{4.8}$$

where $M^1(t)$ is defined as

$$M^1(t) = u^1 - u^1(0) + \chi \int_0^t u^1(s)ds + \int_0^t B(u^1(s), u^1(s))ds - \int_0^t f(s)ds$$

P-a.s. in $C([0,T];D(A^{-\beta/2}))$.
From (4.7) and (4.8), with $v, z \in D(A^{-\beta/2})$, we have that $A^{-\beta/2}M^1(t)$ is a square integrable martingale in H with respect to the filtration

$$\sigma \left\{ u^1(s), s \le t \right\},$$

with quadratic variation

$$<< A^{-\beta/2}M^1 >>_t = \int_0^t A^{-\beta/2}G(u^1)G(u^1)^* A^{-\beta/2} ds.$$

We conclude by a representation theorem (see [9] p233).

5. Appendix

For any Progressively measurable process $f \in L^p(\Omega \times [0,T]; L_2(K,H))$ denote by $I(f)$ the Ito integral defined as

$$I(f)(t) = \int_0^t f(s)dW(s), \ t \in [0,T].$$

$I(f)$ is a progressively measurable process in $L^p(\Omega \times [0,T]; H)$.

Lemma 5.1. *Let $p \geq 2$ and $\gamma < 1/2$ be given. Then for any progressively measurable process $f \in L^p(\Omega \times [0,T]; L_2(K,H))$, we have*

$$I(f) \in L^p(\Omega; W^{\gamma,p}(0,T;H))$$

and there exists a constant $C(p,\gamma) > 0$ independent of f such that

$$\mathbf{E} \parallel I(f) \parallel^p_{W^{\gamma,p}(0,T;H)} \leq C(p,\gamma)\mathbf{E} \int_0^T \parallel f \parallel^p_{L_2(K;H)} dt.$$

Proof. see [11]. □

Theorem 5.2. *Let $B_0 \subset B \subset B_1$ be Banach spaces, B_0 and B_1 reflexive with compact embedding of B_0 in B_1. Let $p \in (1,\infty)$ and $\gamma \in (0,1)$ be given. Let X be the space*

$$X = L^p(0,T;B_0) \cap W^{\gamma,p}(0,T;B_1)$$

endowed with the natural norm. Then the embedding of X in $L^p(0,T;B)$ is compact.

Theorem 5.3. *Let B_1 and $\tilde{B}$ two Banach spaces such that $B_1 \subset \tilde{B}$ with compact embedding. If the real numbers $\gamma \in (0,1)$ and $p > 1$ satisfy*

$$\gamma p > 1$$

then the space $W^{\gamma,p}(0,T;B_1)$ is compactly embedded into $C([0,T];\tilde{B})$.

Proof. see [11]. □

References

[1] R. A. Adams, J. J. F. Fournier, *Sobolev spaces*, Pure and Applied Mathematics Series, 2nd edition, Academic Press, 2003.

[2] C. Bardos, *Éxistence et unicité de la solution de l'équation d'Euler en dimensions deux*, Jour. Math. Anal. Appl. **40**, 1972, 769-780.

[3] H. Bessaih, F. Flandoli, *2-D Euler equations with non regular force*, Non Diff.Eq. Appl, Vol **6**, Issue 1, 1999, pp35-54.

[4] H. Bessaih, *Martingale solutions for stochastic Euler equations*, Stoc. Anal. Appl, Vol 17 number 5, 1999, pp713-727.

[5] Z. Brzeźniak, M. Capiński, F. Flandoli, *Stochastic partial differential equations and turbulence*, Math. Models and Methods in Appl. Sc, **1**, (1991), 41-59.

[6] Z. Brzeźniak, M. Capiński, F. Flandoli, *Stochastic Navier-Stokes equations with multiplicative noise*, Stoc.Anal.Appl, **10**(5), (1992), 523-532.

[7] Z. Brzeźniak, S. Peszat, *Stochastic Euler equations*, Ann. Probab., **29**, (2001), no. 4, 1796-1832.

[8] M. Capiński, N. J. Cutland, *Stochastic Euler equations on the torus*, Ann. Appl. Probab., **9**, no. 3, 688-705, 1999.

[9] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge university press, (1992).

[10] G. Da Prato, J. Zabczyk, Ergodicity for infinite-dimensional systems, Cambridge University Press, Cambridge (1996).

[11] F. Flandoli, D. Gatarek, *Martingale and stationary solutions for stochastic Navier-Stokes equations*, Probab. Theory relat. Fields, **102**, (1995), 367-391.

[12] G. Gallavotti, Ipotesi per una introduzione alla Meccanica Dei Fluidi, Quaderni del Consiglio Nazionale delle Ricerche, Gruppo Nazionale di Fisica Matematica, no 52, (1996).

[13] J.L. Lions, *Équations Différentielles Opérationelles et Problèmes aux limites*, Springer-Verlag, Berlin (1961).

[14] P.L. Lions, *Mathematical Topics in Fluid Mechanics*, Vol 1, Incompressible Models, Oxford Sci. Publ, Oxford (1996).

[15] R. Mikulevicius, G. Valiukevicius, *On stochastic Euler equation in* $\mathbf{R}^d$, Electron. J. Probab., **5**, no. 6, 2000.

[16] W.A. Strauss,*On continuity of functions with values in various Banach spaces*, Pacific journal of Mathematics, Vol **19**, No 3, 1966.

[17] R. Temam, *Navier-Stokes equations*, North-Holland, 1984.

[18] K. Yosida, *Functional Analysis*, Classics in Mathematics, Springer-Verlag, Berlin, 1995.

Acknowledgment

I would like to thank the referee for pointing out a mistake in the first version of the paper and for his helpful comments.

Hakima Bessaih
University of Wyoming
Department of Mathematics
Laramie 82070 WY
USA
e-mail: `bessaih@uwyo.edu`