1 Open and Closed Sets of Real Numbers

The simplest sets of real numbers are the intervals. We define the open interval $(a, b)$ to be the set

$$(a, b) = \{ x : a < x < b \}.$$ 

We also consider the infinite interval $(a, \infty)$,

$$(a, \infty) = \{ x : a < x \},$$

and

$$(-\infty, b) = \{ x : x < b \}.$$ 

Sometimes we write $(-\infty, \infty)$ for the set of all real numbers. We define the closed interval $[a, b]$ to be the set

$$[a, b] = \{ x : a \leq x \leq b \}.$$ 

For closed intervals we take $a$ and $b$ finite but always assume that $a < b$.

The half-open interval $(a, b]$ is defined to be

$$(a, b] = \{ x : a < x \leq b \},$$

and

$$[a, b) = \{ x : a \leq x < b \}.$$ 

A generalization of the notion of an open interval is given by that of an open set.

**Definition 1** A set $O$ of real numbers is called open if for each $x \in O$ there is a $\delta > 0$ such that each $y$ with $|x - y| < \delta$ belongs to $O$.

**Definition 2** A set $O$ of real numbers is called open if for each $x \in O$ there is an open interval $I$ such that $x \in I \subset O$.

**Example 3**

1. The open intervals are examples of open sets.

2. the empty set $\emptyset$ and the set $\mathbb{R}$ of real numbers are open.
We establish some properties of open sets.

**Proposition 4** The intersection $O_1 \cap O_2$ of two open sets $O_1$ and $O_2$ is open.

**Proof.** Let $x \in O_1 \cap O_2$. Since $x \in O_1$ and $O_1$ is open, there is a $\delta_1 > 0$ such that all $y$ with $|x - y| < \delta_1$ belongs to $O_1$. Similarly, there is a $\delta_2 > 0$ such that all $y$ with $|x - y| < \delta_2$ belongs to $O_2$. Take $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$ and if $|x - y| < \delta$, then $y$ belongs to both $O_1$ and $O_2$ which means to $O_1 \cap O_2$. □

**Corollary 5** The intersection of any finite collection of open sets is open.

**Proposition 6** The union of any collection of open sets is open.

**Example 7** It is not true that the intersection of any collection of open sets is open. Take for example $O_n$ to be the open interval $(-\frac{1}{n}, \frac{1}{n})$. Then
\[
\bigcap_{n=1}^{\infty} O_n = \{0\},
\]
and $\{0\}$ is not an open set.

**Proposition 8** Every open set of real numbers is the union of a countable collection of disjoint open intervals.

**Proof.** Since $O$ is open, for each $x \in O$ there is a $y > x$ such that $(x, y) \subset O$ and there is a $z < x$ such that $(z, x) \subset O$. Let
\[
b = \sup\{y : (x, y) \subset O\}, \quad a = \inf\{z : (z, x) \subset O\}.
\]
Then $a < x < b$, and $x \in I_x = (a, b) \subset O$.

**Characterization of the $b$:** $\forall \epsilon > 0, b - \epsilon < x < b$.

**Characterization of the $a$:** $\forall \epsilon > 0, a < x < a + \epsilon$.

Which implies that $a \notin O$ and $b \notin O$.

Consider the collection of open intervals $\{I_x\}, x \in O$. Since each $x \in O$ is in $I_x$ and each $I_x$ is contained in $O$, we have that
\[
O = \bigcup I_x.
\]

Let us prove that the open intervals must be disjoint. In fact, let $(a, b)$ and $(c, d)$ be two intervals in this collection with a point in common. Then, we must have that $c < b$ and $a < d$. Since $c \notin O$ then $c \notin (a, b)$ and we have that $c \leq a$. Since $a \notin O$ then $a \notin (c, d)$ and we have that $a \leq c$. Thus $a = c$. Similarly, we can prove that $b = d$. Thus two different intervals in the collection $\{I_x\}$ must be disjoint. It remains only to prove that this collection is countable: Each open interval contains a rational number (Archimedean Axiom), hence the collection can be put in a one-to-one correspondence with a subset of rational numbers. Thus, this collection is countable. □
Proposition 9 (Lindelöf) Let \( C \) a collection of open sets of real numbers. Then there is a countable subcollection \( \{O_i\} \) of \( C \) such that

\[
\bigcup_{O \in C} O = \bigcup_{i=1}^{\infty} O_i.
\]

Proof. See Royden pp42. ■

We shall also study the notion of a closed set which generalizes the notion of a closed interval.

Definition 10 (Point of closure or limit point) A real number \( x \) is called a point of closure of a set \( E \) if for every \( \delta > 0 \) there is a \( y \in E \) (\( x \neq y \)) such that \( |x - y| < \delta \).

This equivalent to say that \( x \) is a point of closure of \( E \) if every open interval containing \( x \) also contains a point of \( E \), that is \( \forall I_x \ (I_x \cap E) \setminus \{x\} \neq \emptyset \).

We denote the set of points of closure of \( E \) by \( \overline{E} \). Thus \( E \subset \overline{E} \).

Example 11
- \( 2 \) is a point of closure of the interval \( (2, \infty) \) and its closure is equal to \( [2, \infty) \), that is \( (2, \infty) = [2, \infty) \).
- \( 2 \) is not a point of closure of the set \( \{2\} \cup (3, \infty) \).

Proposition 12 If \( A \subset B \) then \( \overline{A} \subset \overline{B} \). And

\[
\overline{A \cup B} = \overline{A} \cup \overline{B}.
\]

Definition 13 (Closed set) A set \( F \) is called closed if \( F = \overline{F} \). (it is sufficient to state that \( F \subset \overline{F} \)).

Example 14 \( [a, b] \) and \( [a, \infty) \) are closed sets.

Proposition 15 For any set \( E \), the set \( \overline{E} \) is closed, that is \( \overline{E} = \overline{E} \).

Proposition 16 The complement of an open set is closed and the complement of a closed set is open.

Proof. \( O \) is open \( \implies \) if \( x \in O \) there exits \( \delta > 0 \) such that if \( |x - y| < \delta \) then \( y \in O \). This means that \( y \notin \overline{O} \), which means that \( x \) cannot be a point of closure of \( \overline{O} \) since there no \( y \in \overline{O} \) with \( |x - y| < \delta \). Thus \( \overline{O} \) contains all its points of closure and therefore is closed. ■

Proposition 17 The union \( F_1 \cup F_2 \) of two closed sets \( F_1 \) and \( F_2 \) is closed.

Proof. \( \overline{F_1 \cup F_2} = \overline{F_1} \cup \overline{F_2} = F_1 \cup F_2 \). ■

Proposition 18 The intersection of any collection of closed sets is closed.
Proof. Using the De Morgan’s Law we have that $\bigcap_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} \bar{F}_i$ is open. Hence $\bigcap_{i=1}^{\infty} F_i$ is closed. □

We say that a collection $\mathcal{C}$ of sets covers a set $F$ if $F \subseteq \bigcup \{O : O \in \mathcal{C}\}$. In this case the collection $\mathcal{C}$ is called a covering of $F$. If each $O$ is open, we call $\mathcal{C}$ an open covering of $F$. If $\mathcal{C}$ contains a finite number of sets, we call $\mathcal{C}$ a finite covering.

Definition 19 (compact set) A set $F$ is called compact if every open covering of $F$ can be reduced to a finite open covering of $F$, that is if $\mathcal{C}$ is a collection of open sets such that $F \subseteq \bigcup \{O : O \in \mathcal{C}\}$, then there is a finite collection $\{O_1, \ldots, O_n\}$ of sets in $\mathcal{C}$ such that $F \subseteq \bigcup_{i=1}^{n} O_i$.

Theorem 20 (Heine-Borel) A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.

Example 21

• $[0, 2]$ is compact.

• $[2, \infty)$ is not compact because it is not bounded.

• Cantor sets. Cantor sets are fascinating examples of compact sets. Here is how to construct the standard Cantor set: Start with the unit interval $[0,1]$ and remove its open middle third, $(1/3,2/3)$. Then remove the open middle third from the remaining two intervals, and so on. This gives you a nested sequence

$C^0 \supset C^1 \supset C^2 \ldots$

where $C^0 = [0,1]$, $C^1$ is the union of two intervals $[0, 1/3]$ and $[2/3, 1]$, $C^2$ is the union of four intervals $[0, 1/9], [2/9, 1/3], [2/3, 7/9]$ and $[8/9, 1]$, $C^3$ is the union of eight intervals and so on. In general $C^n$ is the union of $2^n$ intervals, each of length $1/3^n$. The Cantor set denoted by $C$ is given by

$C := \bigcap C^n$.

Clearly $C$ contains the endpoints of each of the intervals comprising $C^n$. Actually, it contains uncountably many more points than these endpoints. $C$ is uncountable and compact.

2 Continuous functions

Let $f$ be a real-valued function whose domain of definition is a set $E$ of real numbers.

Definition 22 We say that $f$ is continuous at the point $x \in E$ if given $\epsilon > 0$, there is a $\delta > 0$ such that for all $y \in E$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \epsilon$. 

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Definition 23 The function $f$ is said to be continuous on a $A \subset \mathbb{R}$ if it is continuous at each point of $A$.

Proposition 24 Let $f$ be continuous on a closed and bounded set $F \subset \mathbb{R}$. Then $f$ is bounded on $F$ and assumes its maximum and minimum on $F$; that is, there are points $x_1, x_2 \in F$ such that

$$f(x_1) \leq f(x) \leq f(x_2), \quad \forall x \in F.$$ 

Proof. $F$ is compact this means that from each open covering of $F$ we can extract a finite open sub-covering of $F$.

Let us first prove that $f$ is bounded on $F$. Since $f$ is continuous on $F$, this means that for each

$$\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in F, \quad |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$ 

For $\epsilon = 1$, for each $x \in F$, there is an open interval $I_x \ni x$ such that $|f(y)| < 1 + |f(x)|$ for $y \in I_x \cap F$.

On the other hand, the collection $\{I_x : x \in F\}$ is an open covering of $F$ which is compact so we can extract a finite subcollection $\{I_{x_1}, \ldots, I_{x_n}\}$ such that $F \subset \bigcup_{i=1}^{n} I_{x_i}$. Let $M = 1 + \max \{ |f(x_1)|, \ldots, |f(x_n)| \}$.

If $y \in F$ then there exits $k$ in $[1, n]$ such that $y$ in $I_{x_k}$. Hence $|f(y)| < 1 + |f(x_k)| \leq M$. Hence $f$ is bounded on $F$.

Now, we have to prove that $f$ assumes its maximum and minimum on $F$. Let $M := \sup \{ f(x) : x \in F \}$. Then because $f$ is bounded $M < \infty$. We have to prove that there exists $x_1 \in F$ such that $M = f(x_1)$. Suppose not, then $f(x) < M$ $\forall x \in F$. By the continuity of $f$, for all $\epsilon > 0$ there is an open interval $I_x \ni x$ such that $f(y) < f(x) + \epsilon$ $\forall y \in I_x \cap F$. Let us take $\epsilon = \frac{1}{2}(M - f(x))$ then $f(y) < \frac{1}{2}(f(x) + M) \forall x \in I_x \cap F$. Let $a := \max \{ f(x_1), \ldots, f(x_n) \} < M$.

Now using the fact that $F$ is compact, each $y \in F$ belongs to some $I_{x_k}$ and $f(y) < \frac{1}{2}(f(x_k) + M) \leq \frac{1}{2}(M + a)$. This means that $\frac{1}{2}(M + a)$ is a bound for $f$. Which contradicts the fact that $M$ is the supremum. Hence there is $x_1 \in F$ such that $f(x_1) = M$. Use the similar Proof for the minimum. 

Proposition 25 Let $f$ be a real-valued function on $\mathbb{R}$. Then $f$ is continuous if and only if for each open set $O$ of real numbers $f^{-1}(O)$ is an open set.

Proof. $(\implies)$: Let $\epsilon > 0$. The interval $I := (f(x) - \epsilon, f(x) + \epsilon)$ is an open interval and so its inverse image must be open. Which means that $x \in f^{-1}(I)$, because

$$f^{-1}(I) := \{ x \in \mathbb{R} : f(x) \in I \}.$$ 

Hence there is a $\delta > 0$ such that $(x - \delta, x + \delta) \subset f^{-1}(I)$, which means that if $y \in (x - \delta, x + \delta)$ then $y \in f^{-1}(I)$. In other words, if $|x - y| < \delta$ then
$f(y) \in (f(x) - \epsilon, f(x) + \epsilon)$; that is $|f(x) - f(y)| < \epsilon$. Hence $f$ is continuous, since $x$ is arbitrary.

$(\Leftarrow)$: See Royden. ■

**Definition 26** (Uniform continuity) A real-valued function $f$ is defined on a set $E$ is said to be uniformly continuous on $E$ if given $\epsilon > 0$, there is a $\delta > 0$ such that for all $x$ and $y$ in $E$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

**Proposition 27** If a real-valued function $f$ is defined and continuous on a compact set $F \subset \mathbb{R}$, then it is uniformly continuous on $F$.

**Proof.** See Royden pp 48. ■

**Definition 28** (Pointwise convergence of sequence of functions) A sequence of functions $\{f_n\}$ defined on a set $E$ is said to converge pointwise on $E$ to a function $f$ if for every $x \in E$, \[
\lim_{n \to \infty} f_n(x) = f(x); \]
that is

\[
\forall x \in E, \ \forall \epsilon > 0, \ \exists N \in \mathbb{N}, \ \forall n \geq N, \ \ |f(x) - f_n(x)| < \epsilon.
\]
Here $N$ can depend on $\epsilon$ and $x$.

**Definition 29** (Uniform convergence of sequence of functions) A sequence of functions $\{f_n\}$ converges uniformly to $f$ on $E$ if

\[
\forall \epsilon > 0, \ \exists N(\epsilon), \ \ \forall n \geq N(\epsilon), \ |f(x) - f_n(x)| < \epsilon, \ \text{for all} \ x \in E.
\]