1. Suppose $f \geq 0$ and $f$ is integrable. If $\alpha > 0$ and 

$$E_\alpha = \{x : f(x) > \alpha\}$$

prove that

$$m(E_\alpha) \leq \frac{1}{\alpha} \int f.$$  

2. Suppose that $f$ is integrable on $(-\pi, \pi]$ and extended to $\mathbb{R}$ by making it periodic of period $2\pi$. Show that

$$\int_{-\pi}^{\pi} f(x)dx = \int_I f(x)dx,$$

where $I$ is any interval in $\mathbb{R}$ of length $2\pi$.

**Hint:** $I$ is contained in two consecutive intervals of the form $(k\pi, (k + 2)\pi]$.

3. Investigate the convergence of

$$\int_a^\infty \frac{n^2xe^{-n^2x^2}}{1 + x^2}dx$$

for $a > 0$, and for $a = 0$.

4. Let $f$ be real-valued function

$$f(x) = \begin{cases} 
\frac{(-1)^n}{n+1} & x \in [n, n + 1), \quad n \geq 0 \\
0 & x < 0
\end{cases}$$

(a) Prove that the improper Riemann integral of $f$ exists.

(b) Prove that $f$ is not Lebesgue integrable on $\mathbb{R}$.

5. Let $f$ be an integrable function on $[0, 1]$. Show that $x^n f(x)$ is integrable on $[0, 1]$, for $n = 1, 2, \ldots$ and compute

$$\lim_{n \to \infty} \int_{[0,1]} x^n f(x)dx.$$
6. Let $I = [0, 1]$ and suppose that $\{f_n\}_n$ is a sequence of real-valued measurable functions defined on $I$ such that
\[ \lim_{n \to \infty} f_n = f \text{ a.e.} \]
Prove that
\[(a) \quad \lim_{n \to \infty} \int_I |f_n(x)|e^{-|f_n(x)|}dx = \int_I |f(x)|e^{-|f(x)|}dx.\]
\[(b) \quad \int_I \sum_n (f_n(x))^2 \, dx = \sum_n \int_I (f_n(x))^2 \, dx.\]

7. Prove that for any integrable functions $f$ and $g$ on a measurable set $E$, their sum is integrable and
\[ \int_E (f + g)dm = \int_E f dm + \int_E g dm. \]

8. Let $f(x) = \frac{\sin x}{x}$, $x \neq 0$. Prove that $\int_{\mathbb{R}} f(x)dx < \infty$, but that $f \notin L^1$.

9. Are the following sequences Cauchy in $L^1(0, \infty)$.
   \[(a) \quad f_n(x) = \frac{1}{\pi} \chi_{[n,n+1]}(x) \]
   \[(b) \quad f_n(x) = \chi_{[n,n+1]}(x) \]
   \[(c) \quad f_n(x) = \frac{1}{\pi} \chi_{[0,n]}(x) \]
   \[(d) \quad f_n(x) = \frac{1}{\pi^2} \chi_{[0,n]}(x) \]

10. Let $I = [0, 1]^2$ and let us define a function $f$ on $I$ by
\[ f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}, \quad x, y \in I. \]
Prove that $f \notin I$. 

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