Find the range of each function $f : \mathbb{R} \to \mathbb{R}$.

(a) $f(x) = x^2 + 1$
(b) $f(x) = (x + 3)^2 - 5$
(c) $f(x) = x^2 + 4x + 1$
(d) $f(x) = 2 \cos 3x$

7.4 (a) Let $S$ be the set of all circles in the plane. Define $f : S \to [0, \infty)$ by $f(C) =$ the area of $C$, for all $C \in S$. Is $f$ injective? Is $f$ surjective?
(b) Let $T$ be the set of all circles in the plane that are centered at the origin. Define $g : T \to [0, \infty)$ by $g(C) =$ the area of $C$, for all $C \in T$. Is $g$ injective? Is $g$ surjective?

7.5 Let $f$ and $g$ be functions. Prove that $f = g$ iff $\text{dom } f = \text{dom } g$ and for every $x \in \text{dom } f$, $f(x) = g(x)$.

7.6 Suppose that $f : A \to B$, $g : B \to C$, and $h : C \to D$. Prove that $h \circ (g \circ f) = (h \circ g) \circ f$.

7.7 Let $f : A \to B$ and $g : B \to C$. Using the ordered pair definition of the composition $g \circ f$, prove that $g \circ f$ is a function and that $g \circ f : A \to C$.

7.8 In each part, find a function $f : \mathbb{N} \to \mathbb{N}$ that has the desired properties.
(a) surjective, but not injective
(b) injective, but not surjective
(c) neither surjective nor injective
(d) bijective

7.9 (a) Suppose that $A$ has exactly two elements and $B$ has exactly three. How many different functions are there from $A$ to $B$? How many of these are injective? How many are surjective?
(b) Suppose that $A$ has exactly three elements and $B$ has exactly two.
How many different functions are there from $A$ to $B$? How many of these are injective? How many are surjective?
(c) Suppose that $A$ has exactly $m$ elements and $B$ has exactly $n$ (where $m, n \in \mathbb{N}$). How many different functions are there from $A$ to $B$?

7.10 Find examples to show that equality does not hold in parts (a), (b), and (c) of Theorem 7.14. For instance, in part (a) find specific sets $A$, $B$, and $C$, with $C \subseteq A$, and a specific function $f : A \to B$ such that $C \neq f^{-1}[f(C)]$.

7.11 Prove parts (a), (b), (d), (e), and (g) of Theorem 7.14.

7.12 Prove parts (a) and (b) of Theorem 7.16.

7.13 Prove Theorem 7.18(b). That is, suppose that $f : A \to B$ and $g : B \to C$ are both injective. Prove that $g \circ f : A \to C$ is injective.

7.14 Suppose that $f : A \to B$ and suppose that $C \subseteq A$ and $D \subseteq B$.
(a) Prove or give a counterexample: $f(C) \subseteq D$ if $C \subseteq f^{-1}(D)$.
ANSWERS TO PRACTICE PROBLEMS

12.4 Any real number \( x \) such that \( x^2 \geq 2 \) is an upper bound for \( T \). The smallest of these upper bounds is \( \sqrt{2} \), but since \( \sqrt{2} \not\in \mathbb{Q} \), set \( T \) has no maximum. The minimum of \( T \) is 0. Any real \( x \) such that \( x \leq 0 \) is a lower bound.

12.6 \( m = \inf S \) iff (i) \( m \leq s \), for all \( s \in S \), and (ii) if \( m' > m \), then there exists \( s' \in S \) such that \( s' < m' \).

12.13 Since \( x \) is rational and \( x \neq 0 \), we have \( x = m/n \) for some nonzero integers \( m \) and \( n \). If \( xy \) were rational, then we could write \( xy = p/q \) for some \( p, q \in \mathbb{Z} \). But then

\[
y = \frac{xy}{x} = \frac{p/q}{m/n} = \frac{pn}{mq},
\]

so \( y \) would have to be rational too, a contradiction.

EXERCISES

12.1 Mark each statement True or False. Justify each answer.
(a) If a nonempty subset of \( \mathbb{R} \) has an upper bound, then it has a least upper bound.
(b) If a nonempty subset of \( \mathbb{R} \) has an infimum, then it is bounded.
(c) Every nonempty bounded subset of \( \mathbb{R} \) has a maximum and a minimum.
(d) If \( m \) is an upper bound for \( S \) and \( m' < m \), then \( m' \) is not an upper bound for \( S \).
(e) If \( m = \inf S \) and \( m' < m \), then \( m' \) is a lower bound for \( S \).

12.2 Mark each statement True or False. Justify each answer.
(a) For each real number \( x \) and each \( \epsilon > 0 \), there exists \( n \in \mathbb{N} \) such that \( nx > x \).
(b) If \( x \) and \( y \) are irrational, then \( xy \) is irrational.
(c) Between any two unequal rational numbers there is an irrational number.
(d) Between any two unequal irrational numbers there is a rational number.
(e) The rational and irrational numbers alternate, one then the other.

12.3 For each subset of \( \mathbb{R} \), give its supremum and its maximum, if they exist. Otherwise, write “none.”
(a) \( \{1, 3\} \)  (b) \( \{\pi, 3\} \)
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(c) \[0, 4\]  
(d) (0, 4)

(e) \[\left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \]  
(f) \[\left\{ -\frac{1}{n} : n \in \mathbb{N} \right\} \]

(g) \[\left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\} \]  
(h) \[\left\{ (-1)^n \left( 1 + \frac{1}{n} \right) : n \in \mathbb{N} \right\} \]

(i) \[\left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\} \]  
(j) (-\infty, 4)

(k) \[\bigcap_{n=1}^{\infty} \left( 1 - \frac{1}{n}, 1 + \frac{1}{n} \right) \]  
(l) \(\bigcup_{n=1}^{\infty} \left[ \frac{1}{n}, 2 - \frac{1}{n} \right] \]

(m) \(\{ r \in \mathbb{Q} : r < 4 \} \)  
(n) \(\{ r \in \mathbb{Q} : r^3 \leq 5 \} \)

12.4 Repeat Exercise 12.3 for the infimum and the minimum of each set.

12.5 Let \( S \) be a nonempty bounded subset of \( \mathbb{R} \) and let \( m = \sup S \). Prove that \( m \in S \) if \( m = \max S \).

12.6 Let \( S \) be a nonempty bounded subset of \( \mathbb{R} \). Prove that \( \sup S \) is unique.

*12.7 Let \( S \) be a nonempty bounded subset of \( \mathbb{R} \) and let \( k \in \mathbb{R} \). Define \( kS = \{ ks : s \in S \} \). Prove the following:
(a) If \( k \geq 0 \), then \( \sup(kS) = k \cdot \sup S \) and \( \inf(kS) = k \cdot \inf S \).
(b) If \( k < 0 \), then \( \sup(kS) = k \cdot \inf S \) and \( \inf(kS) = k \cdot \sup S \).

12.8 Let \( S \) and \( T \) be nonempty bounded subsets of \( \mathbb{R} \) with \( S \subseteq T \). Prove that \( \inf T \leq \inf S \leq \sup S \leq \sup T \).

12.9 (a) Prove: If \( x > 0 \), then there exists \( n \in \mathbb{N} \) such that \( n - 1 \leq y < n \).
(b) Prove that the \( n \) in part (a) is unique.

12.10 (a) Prove: If \( x \) and \( y \) are real numbers with \( x < y \), then there are infinitely many rational numbers in the interval \([x, y] \).
(b) Repeat part (a) for irrational numbers.

12.11 Let \( y \) be a positive real number. Prove that for every \( n \in \mathbb{N} \) there exists a unique positive real number \( x \) such that \( x^n = y \).

*12.12 Let \( D \) be a nonempty set and suppose that \( f : D \to \mathbb{R} \) and \( g : D \to \mathbb{R} \). Define the function \( f + g : D \to \mathbb{R} \) by \((f + g)(x) = f(x) + g(x)\).
(a) If \( f(D) \) and \( g(D) \) are bounded above, then prove that \((f + g)(D)\) is bounded above and \( \sup [(f + g)(D)] \leq \sup f(D) + \sup g(D) \).
(b) Find an example to show that a strict inequality in part (a) may occur.
(c) State and prove the analog of part (a) for infima.

12.13 Let \( x \in \mathbb{R} \). Prove that \( x = \sup \{ q \in \mathbb{Q} : q < x \} \).
Mark each statement True or False. Justify each answer.

(a) Every sequence has a convergent subsequence.
(b) The set of subsequential limits of a bounded sequence is always nonempty.
(c) \((s_n)\) converges to \(s\) iff \(\lim \inf s_n = \lim \sup s_n = s\).
(d) If \((s_n)\) is unbounded above, then \(\lim \inf s_n = \lim \sup s_n = +\infty\).
(e) Let \((s_n)\) be a bounded sequence and let \(m = \lim \sup s_n\). Then for every \(\varepsilon > 0\) there are infinitely many terms in the sequence greater than \(m - \varepsilon\).

For each sequence, find the set \(S\) of subsequential limits, the limit superior, and the limit inferior.

(a) \(s_n = (-1)^n\)
(b) \((t_n) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots\right)\)
(c) \(u_n = n^2 \cdot (-1 + (-1)^n)\)
(d) \(v_n = n \cdot \sin \frac{\pi n}{2}\)

For each sequence, find the set \(S\) of subsequential limits, the limit superior, and the limit inferior.

(a) \(w_n = \frac{(-1)^n}{n}\)
(b) \((x_n) = (0, 1, 2, 0, 1, 3, 0, 1, 4, \ldots)\)
(c) \(y_n = n(2 + (-1)^n)\)
(d) \(z_n = (-n)^n\)

Use Exercise 18.10 to find the limit of each sequence.

(a) \(s_n = \left(1 + \frac{1}{2n}\right)^{2n}\)
(b) \(s_n = \left(1 + \frac{1}{n}\right)^{2n}\)
(c) \(s_n = \left(1 + \frac{1}{n}\right)^n\)
(d) \(s_n = \left(1 + \frac{1}{n}\right)^n\)
(e) \(s_n = \left(1 + \frac{1}{2n}\right)^n\)
(f) \(s_n = \left(\frac{n+2}{n+1}\right)^{n+3}\)

If \((s_n)\) is a subsequence of \((t_n)\) and \((t_n)\) is a subsequence of \((s_n)\), can we conclude that \((s_n) = (t_n)\)? Prove or give a counterexample.

Let \((s_n)\) be a bounded sequence and suppose that \(\lim \inf s_n = \lim \sup s_n = s\). Prove that \((s_n)\) is convergent and that \(\lim s_n = s\).