Partial credit will be awarded for your answers, so it is to your advantage to explain your reasoning and what theorems you are using when you write your solutions. Please answer the questions in the space provided and show your computations.

Good luck!

Name: ____________________________
I. (15 points) Compute the lim inf and lim sup of the following sequence

$$x_n = \sin \frac{n\pi}{2} + \frac{(-1)^n}{n}, \quad n \in \mathbb{N}$$
II. (15 points) Define $f : \mathbb{R} \rightarrow \mathbb{R}$ as follows:

$$f(x) = \begin{cases} 
  5x, & \text{if } x \text{ is rational} \\
  x^2 + 6, & \text{if } x \text{ is irrational}
\end{cases}$$

1. Prove that $f$ is discontinuous at 1 and continuous at 2.

2. Are there other points besides 2 at which $f$ is continuous?
III. (10 points) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function and let $k \in \mathbb{R}$. Prove that the set

$$f^{-1}(\{k\})$$

is closed.

**Hint:** Recall that $f^{-1}(\{k\}) = \{x \in \mathbb{R} : f(x) = k\}$
VI. **(10 points)** Using the $\epsilon - \delta$ property, prove that the function

$$f(x) = \frac{x - 1}{x + 1},$$

is uniformly continuous on $[0, \infty)$. 
V. (10 points) Use the mean value theorem to establish the following inequality

\[ |\cos x - \cos y| \leq |x - y|, \quad \text{for } x, y \in \mathbb{R}. \]
VI. (10 points) Determine whether the following series converges or diverges. Justify your answer.

\[ \sum 2^n e^{-n} \]
VII. **(10 points)** Find the values of $x$ for which the following power series is convergent

$$
\sum (n3^{-n})(x - 2)^n
$$
VIII. (20 points) Let \( f(x) = \ln(1 + x), \ x \in (-1, 1). \)

1. Write out the Taylor polynomial of order \( n \) for \( f \) at the point \( x_0 = 0 \)
2. What is the Taylor remainder \( R_n(x) \)?
3. Prove that \( f \) can be written in terms of a power series on \((-1, 1)\).