1. (ex 10 pp44) Let $X$ be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p, q) = \begin{cases} 
1 & \text{if } p \neq q \\
0 & \text{if } p = q 
\end{cases}$$

(a) Prove that this is a metric (distance).

(b) Which subsets of the resulting metric space are open?

(c) Which subsets of the resulting metric space are closed?

(d) Which subsets of the resulting metric space are compact?

**Solution:**

(a) i. By definition of the metric $d(p, q) \geq 0$ $\forall p, q \in X$. Moreover, $d(p, q) = 0$ implies that $p = q$.

ii. It is also clear from the definition of the metric $d$ that $d(p, q) = d(q, p)$ $\forall p, q \in X$.

iii. let $p, q \in X$ such that $p \neq q$, then $d(p, q) = 1$. Then for any other $r \in X$ such that $r \neq p, r \neq q$, then

$$1 = d(p, q) \leq d(p, r) + d(r, q) = 2.$$

Similarly, one can look at all the different cases $p = r, p \neq r$.

Hence, we conclude that $d$ is a metric in $X$.

(b) Let $x \in X$ and let us define the open ball in $X$ of center $x$ and radius $r$, $B(x, r)$. Hence,

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

If $r \leq 1$, then $B(x, r) = \{x\}$, hence $\{x\}$ is open for any $x \in X$. Since, any countable union of open sets is open, we deduce that any subset of $X$ is open.

(c) Since, any subset of $X$ is open and since any complement of an open set is closed, we deduce that any subset of $X$ is closed.

(d) Let us denote by $E$ any nonempty subset of $X$. By definition of the metric $d$, for any arbitrary points $x, y \in E$, $d(x, y) \leq 1$. Hence any subset $E$ of $X$ is bounded. Since it is also closed then it will be compact by the Heine-Borel Theorem (Heine-Borel Theorem is true only when $X \subset \mathbb{R}^n$).
2. *(ex 11 pp44)* For \(x\) and \(y\) in \(\mathbb{R}^1\), define

(a) \(d_1(x, y) = |x^2 - y^2|\),
(b) \(d_2(x, y) = \sqrt{|x - y|}\),
(c) \(d_3(x, y) = |x - 2y|\),
(d) \(d_4(x, y) = \frac{|x-y|}{1+|x-y|}\),

Determine for each of these, whether it is a metric (distance) or not.

**Solution:**

(a) \(d_1\) is not a metric because it violates the first property, that is, there exists two points \(x = 1 \neq y = -1\) such that \(d_1(x, y) = 0\).

(b) \(d_2\) is a metric and it’s easy to verify that the three properties of a metric are satisfied.

(c) \(d_3\) is not a metric because it violates the first property, that is, there exists two points \(x = 1 \neq y = 2\) such that \(d_3(x, y) = 0\).

(d) \(d_4\) is a metric. It is easy to verify the first two properties of a metric. Let us prove the triangle property. Let \(x, y, z \in \mathbb{R}^1\), then using the triangle property for the absolute value, we get that

\[
|x - y| \leq |x - z| + |z - y|.
\]

Now, adding the same term of both sides of the inequality gives

\[
|x - y| + |x - y|(|x - z| + |y - z|) \leq |x - z| + |z - y| + |x - y|(|x - z| + |y - z|) + 2|x - y|(|x - z| + |y - z|).
\]

Again, adding to the previous inequality the term \(|x - y||x - z||y - z|\) yields

\[
|x - y| + |x - y|(|x - z| + |y - z|) + |x - y||x - z||y - z| \leq |x - z| + |z - y| + |x - y|(|x - z| + |y - z|) + 2|x - y|(|x - z| + |y - z|) + 2|x - y||x - z||y - z|.
\]

Hence,

The previous expressions in the above inequality can be rewritten differently by putting \(|x - y|, |x - z|, |y - z|\) in common factor then we obtain

\[
|x - y|(1 + |x - z|)(1 + |y - z|) \leq \\
|x - z|(1 + |y - z|)(1 + |x - y|) + |y - z|(1 + |y - z|)(1 + |x - y|).
\]

Hence, dividing both sides by \((1 + |x - z|)(1 + |y - z|)\)

\[
|x - y| \leq \frac{|x - z|(1 + |y - z|)(1 + |x - y|)}{(1 + |x - z|)(1 + |y - z|)} + \frac{|y - z|(1 + |y - z|)(1 + |x - y|)}{(1 + |x - z|)(1 + |y - z|)}
\]
Now, dividing both sides by \((1 + |x - y|)\), we get that
\[
\frac{|x - y|}{(1 + |x - y|)} \leq \frac{|x - z|}{(1 + |x - z|)} + \frac{|y - z|}{(1 + |y - z|)}
\]
and this implies that
\[
d_4(x, y) \leq d_4(x, z) + d_4(z, y).
\]

3. (ex 12 pp44) Let \(K \subset \mathbb{R}\) consist of 0 and the numbers \(\frac{1}{n}\) for \(n = 1, 2, 3, \ldots\). Prove that \(K\) is compact directly from the definition (without using the Heine-Borel theorem).

**Solution:** The set \(K\) is given by
\[
K = \left\{0, \frac{1}{n}, \ n = 1, 2, 3, \ldots\right\}.
\]

Let \(\{G_\alpha\}_\alpha\) be an open cover of \(K\) then \(K \subset \bigcup_\alpha G_\alpha\). In particular, \(0 \in \bigcup_\alpha G_\alpha\) and \(\frac{1}{n} \in \bigcup_\alpha G_\alpha, \ n \in \mathbb{N}\). Now, \(0 \in \bigcup_\alpha G_\alpha\) implies that there is \(\alpha_0\) such that \(0 \in G_{\alpha_0}\) which is open. Hence 0 is an interior point for \(G_{\alpha_0}\). Hence, there exists an open ball centered in 0 that is included in \(G_{\alpha_0}\), that is
\[
\exists r > 0, \ B(0, r) \subset G_{\alpha_0}.
\]

This implies that, we can choose \(N\) such that \(Nr > 1\) (By the Archimedian theorem). Hence, for all \(n \geq N, \ \frac{1}{n} \in B(0, r) \subset G_{\alpha_0}\).

On the other side, we know that \(\frac{1}{n} \in \bigcup_\alpha G_\alpha, \ n \in \mathbb{N}\), hence \(\frac{1}{n} \in \bigcup_\alpha G_\alpha, \ n < N\) (the remaining points). Since, this is a finite number of points, we can find a finite number of \(G_{\alpha_n}, \ n < N\) that contain the remaining points. Hence
\[
K \subset G_{\alpha_0} \bigcup \left(\bigcup_{k=1}^{N-1} G_{\alpha_n}\right).
\]
This concludes the proof.