Instructions: Your solutions must appear in an organized and legible format to be given full consideration.

1. Prove that $f(x) = \frac{1}{x}$ is uniformly continuous on $[2, \infty)$ directly verifying the $\epsilon - \delta$ property.

2. A point $x$ is said to be a fixed point of $f$ if $f(x) = x$. Let $f : [a, b] \rightarrow [a, b]$, be a continuous function. Prove that $f$ has a fixed point in $[a, b]$.

3. Suppose that $f : [a, b] \rightarrow \mathbb{R}$ and $g : [a, b] \rightarrow \mathbb{R}$ are continuous functions such that $f(a) \leq g(a)$ and $f(b) \geq g(b)$. Prove that $f(c) = g(c)$ for some $c \in [a, b]$.

4. (Optional) Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function such that $f(x + y) = f(x) + f(y)$, for all $x, y \in \mathbb{R}$.

   (i) Prove that $f$ has a limit at 0 iff $f$ has a limit at every point $c \in \mathbb{R}$.

   **Hint:** First show that $f(nx) = nf(x)$ for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$, so that $f(y/n) = f(y)/n$ for all $n \in \mathbb{N}$ and all $y \in \mathbb{R}$. Then show that if $\lim_{x \to 0} f(x)$ exists, it is equal to 0. Also show that $f(x) = f(x-c) + f(c)$, so that $f(x) - f(c) = f(x-c)$, for all $x, c \in \mathbb{R}$

   (ii) Prove that there exists $k \in \mathbb{R}$ such that $f(x) = kx$ for every $x \in \mathbb{R}$.

   **Hint:** Show that for any rational number $m/n$, $f(m/n) = (m/n)f(1)$. 
