Instructions: Your solutions must appear in an organized and legible format to be given full consideration.

1. Using the $\epsilon/\delta$ definition, prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Solutions $f$ is uniformly continuous on $[0, \infty)$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies that $|\sqrt{x} - \sqrt{y}| < \epsilon$ for all $x, y \in [0, \infty)$.

$\sqrt{x}$ is a continuous function on its domain of definition $[0, \infty)$. Hence, it is uniformly continuous on any compact set as $[0, 2]$. For any arbitrary $\epsilon > 0$. Hence, there is $\delta_1(\epsilon) > 0$.

Let us prove that it is uniformly continuous on $[1, \infty)$. Let $x, y \in [1, \infty)$, hence $x \geq 1$ and $y \geq 1$ which implies that

$$|\sqrt{x} + \sqrt{y}| \geq 2 \implies \frac{1}{\sqrt{x} + \sqrt{y}} \leq \frac{1}{2}$$

$$|\sqrt{x} - \sqrt{y}| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \leq \frac{|x - y|}{2}$$

Hence, it is enough to choose $\delta_2 = \epsilon/2$. Now, let us choose $\delta = \min(\delta_1, \delta_2)$ this implies that $|f(x) - f(y)| < \epsilon$.

2. Suppose $f$ is a uniformly continuous mapping of a metric space $X$ into a metric space $Y$. Prove that $\{f(x_n)\}_n$ is a Cauchy sequence in $Y$ for every Cauchy sequence $\{x_n\}_n$ in $X$.

Solutions Let us denote by $d_X$ the metric of $X$ and by $d_Y$ the metric of $Y$. Now, $f$ is uniformly continuous on $X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $d_X(x, y) < \delta$ implies that $d_Y(f(x), f(y)) < \epsilon$ for all $x, y \in X$.

Since $\{x_n\}_n$ is Cauchy in $X$ there exists $N \in \mathbb{N}$ such that $n \geq m \geq N$ implies that $d_X(x_n, x_m) < \delta$.

Hence, for $n \geq m \geq N$, we have that $d_Y(f(x_n), f(x_m)) < \epsilon$. So $\{f(x_n)\}_n$ is Cauchy in $Y$. 
3. Let $f : I \rightarrow \mathbb{R}$, where $I$ is an interval. We say that $f$ is convex if for every $a, b \in I$ and every $\lambda$ with $0 < \lambda < 1$,

$$f(\lambda b + (1 - \lambda)a) \leq \lambda f(b) + (1 - \lambda)f(a). \quad (1)$$

(a) Prove that for any $x \in (a, b)$, the inequality (1) can be rewritten as follows:

$$f(x) \leq \frac{x - a}{b - a} f(b) + \frac{b - x}{b - a} f(a) \quad (2)$$

(b) Prove that if $f$ is convex on $I$ and $a < b < c$, $a, b, c \in I$, then

$$f(x) \geq \frac{x - a}{b - a} f(b) + \frac{b - x}{b - a} f(a) \quad \text{for } b < x < c \quad (3)$$

and

$$f(x) \geq \frac{c - x}{c - b} f(b) + \frac{x - b}{c - b} f(c) \quad \text{for } a < x < b \quad (4)$$

(c) By using the previous inequalities (2), (3) and (4), prove that if $f$ is convex on an open interval $I$ then it is continuous on $I$.

**Solutions:**

(a) Let us assume that

$$x = \lambda b + (1 - \lambda)a$$

$$= \lambda b + a - \lambda a$$

$$= \lambda(b - a) + a$$

$$\implies$$

$$\lambda = \frac{x - a}{b - a}$$

$$1 - \lambda = 1 - \frac{x - a}{b - a} = \frac{b - x}{b - a}$$

Now, substituting $x = \lambda b + (1 - \lambda)a$, $\lambda = \frac{x - a}{b - a}$ in (1) we get (2).

(b) Since $f$ is convex on the interval $I$ then formula (2) holds for every $a, x, b \in I$ such that $a < x < b$. Now, let us assume that $a < b < x < c$. We can use again (2) for $f$ and $a < b < x$ and we get that

$$f(b) \leq \frac{b - a}{x - a} f(x) + \frac{x - b}{x - a} f(a)$$

hence,

$$f(b) + \frac{b - x}{x - a} f(a) \leq \frac{b - a}{x - a} f(x)$$

Since, $\frac{b - a}{x - a} \geq 0$, we deduce by multiplying by its inverse $\frac{x - a}{b - a}$ that

$$f(x) \geq \frac{x - a}{b - a} f(b) + \frac{b - x}{b - a} f(a)$$
(c) Similarly we use (2) for $f$ and $x < b < c$ and we get that

$$f(b) \leq \frac{b - x}{c - x} f(c) + \frac{c - b}{c - x} f(x)$$

Hence

$$f(b) + \frac{x - b}{c - x} f(c) \leq \frac{c - b}{c - x} f(x)$$

Since $\frac{c - b}{c - x} \geq 0$, then multiplying by its inverse $\frac{c - x}{c - b}$, we get (3).

(d) The function $f$ is continuous on the open interval $I$ if it is continuous on every point of the interval $I$. Let us assume that $b$ be an arbitrary point of the interval $I$ and assume that $a < x < b < c$, hence using the inequalities (2) and (4), we get that

$$\frac{c - x}{c - b} f(b) + \frac{x - b}{c - b} f(c) \leq f(x) \leq \frac{x - a}{b - a} f(b) + \frac{b - x}{b - a} f(a) \quad (5)$$

Let us take the limit of (5) when $x \to b^-$, we get

$$f(b) \leq \lim_{x \to b^-} f(x) \leq f(b)$$

hence, we deduce that $\lim_{x \to b^-} f(x) = f(b)$.

On the other side, let us rewrite the inequality (2) when $b < x < c$, we get

$$f(x) \leq \frac{x - b}{c - b} f(c) + \frac{c - x}{c - b} f(b) \quad (6)$$

combining both (6) and (3), we get that

$$\frac{x - a}{b - a} f(b) + \frac{b - x}{b - a} f(a) \leq f(x) \leq \frac{x - b}{c - b} f(c) + \frac{c - x}{c - b} f(b) \quad (7)$$

Taking the limit of the inequality of (7) when $x \to b^+$, we get

$$f(b) \leq \lim_{x \to b^+} f(x) \leq f(b)$$

hence, we deduce that $\lim_{x \to b^+} f(x) = f(b)$. Thus, $\lim_{x \to b^+} f(x) = f(b) = \lim_{x \to b^-} f(x) = f(b)$. As a consequence, $f$ is continuous at $b$ which is arbitrary in $I$ hence, $f$ is continuous on the whole interval $I$. 