Partial credit will be awarded for your answers, so it is to your advantage to explain your reasoning and what theorems you are using when you write your solutions. Please answer the questions in the space provided and show your computations.

Good luck!
I. (6 points) For \( x \in \mathbb{R} \) and \( y \in \mathbb{R} \), define

\[
d(x, y) = |3x - 2y|.
\]

Prove that \( d \) is not a metric.
II. (10 points) Assume that \( \{p_n\}_n \) and \( \{q_n\}_n \) are Cauchy sequences in a metric space \( X \). Show that the sequence \( \{d(p_n, q_n)\}_n \) converges. Here \( d(\cdot, \cdot) : X \times X \rightarrow \mathbb{R} \)

**Hint:** For any \( n, m \)

\[
d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n)
\]
III. (12 points) Let \((s_n)_n\) a sequence defined by induction as follows:

\[
\begin{cases}
    s_1 &= 1 \\
    s_{n+1} &= \frac{1}{4} (s_n + 5) \\
\end{cases}
\quad n \in \mathbb{N}
\]

1. Prove that \((s_n)_n\) is a monotone and bounded sequence.

2. Find its limit.
VI. (12 points) Find whether or not the series converge. Justify your answer.

1. \[ \sum_{n=1}^{\infty} \frac{\cos(n\pi/2)}{n^2} \]

2. \[ \sum \frac{1}{n^2 \ln\left(\frac{n^2+1}{n^2}\right)} \]
V. \textbf{(10 points)} Suppose $a_n > 0$ and $\sum a_n$ converges. Put

$$r_n = \sum_{k=n}^{\infty} a_k.$$ 

1. Prove that $(r_n)_n$ is a decreasing sequence.

2. If $m < n$, prove that

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}.$$ 

3. Deduce that $\sum \frac{a_n}{r_n}$ diverges.
VI. (Bonus 10 points) Let \((s_n)_n\) a real-valued sequence. Define its arithmetic means \(\sigma_n\) by
\[
\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n + 1}, \quad n = 0, 1, 2, \ldots
\]

1. If \(\lim s_n = s\) prove that \(\lim \sigma_n = s\).

2. Construct a sequence \((s_n)_n\) which does not converge, although \(\lim \sigma_n = 0\)

3. Set \(a_n = s_n - s_{n-1}\) for \(n \geq 1\). Show that
\[
s_n - \sigma_n = \frac{1}{n + 1} \sum_{k=1}^{n} ka_k.
\]

Assume that \(\lim na_n = 0\) and that \((\sigma_n)_n\) converges. Prove that \((s_n)_n\) converges. (This gives a converse to (1) but under the additional assumption that \(na_n \rightarrow 0\)).