Partial credit will be awarded for your answers, so it is to your advantage to explain your reasoning and what theorems you are using when you write your solutions. Please answer the questions in the space provided and show your computations.

Good luck!
I. (10 points) A function $f$ is said to satisfy a Lipschitz condition of order $\alpha$ at $c$ if there exists a positive number $M$ (which may depend on $c$) and a 1-ball $B(c)$ such that
\[ |f(x) - f(c)| < M|x - c|^\alpha \]
whenever $x \in B(c)$, $x \neq c$.

1. Show that a function which satisfies a Lipschitz condition of order $\alpha$ is continuous at $c$ if $\alpha > 0$ and has a derivative if $\alpha > 1$.

2. Give an example of a function satisfying a Lipschitz condition of order 1 at $c$ for which $f'(c)$ does not exist.

Solutions:

1. $f$ is continuous at $c$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$. Hence using the assumption and if $\alpha > 0$, we get that
\[ |f(x) - f(c)| \leq M|x - c|^\alpha \]
\[ < M\delta^\alpha < \epsilon. \]
Then, it is enough to choose $\delta < (\frac{\epsilon}{M})^{1/\alpha}$.

For the derivative at $c$. Let us define the increment of the function $f$ at $c$:
\[ \frac{|f(x) - f(c)|}{|x - c|} \leq \frac{M|x - c|^\alpha}{|x - c|} = M|x - c|^{\alpha - 1}. \]
Now, if we take the limit when $x \to c$ on the previous inequality and if $\alpha > 1$, we get that:
\[ \lim_{x \to c} \frac{|f(x) - f(c)|}{|x - c|} \leq \lim \frac{M|x - c|^{\alpha - 1}}{x - c} = 0. \]
Which implies that $\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = 0$, hence the derivative at $c$ exists.

2. Let us define the following function
\[ f(x) = \begin{cases} x \sin \left( \frac{1}{x} \right) & x \neq 0 \\ 0 & x = 0. \end{cases} \]
The derivative of $f$ at 0 does not exists but it is Lipschitz. Indeed:
\[ \frac{f(x) - f(0)}{x} = \frac{x \sin \left( \frac{1}{x} \right)}{x} = \sin \left( \frac{1}{x} \right) \]
which does not converge when $x \to 0$. Hence $f'(0)$ does not exists.

However,
\[ |f(x) - f(0)| = \left| x \sin \left( \frac{1}{x} \right) \right| \leq |x|. \]
II. (10 points) Let

\[ f(x) = \begin{cases} 
  x + 2x^2 \sin \left( \frac{1}{x} \right) & x \neq 0 \\
  0 & x = 0.
\end{cases} \]

1. Show that \( f \) is differentiable on \( \mathbb{R} \).

2. Compute the derivative of \( f \) on \( \mathbb{R} \).

Solutions:

1. If \( x \neq 0 \), the function \( f \) is differentiable as the sum, composition and product of differentiable functions:

\[
\begin{align*}
  x & \rightarrow x^2 \\
  x & \rightarrow \sin x \\
  x & \rightarrow 1/x \\
  x & \rightarrow x
\end{align*}
\]

If \( x = 0 \), then

\[
\lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{x + 2x^2 \sin \left( \frac{1}{x} \right)}{x} = \lim_{x \to 0} \frac{1}{1 + 2x \sin \left( \frac{1}{x} \right)} = 1.
\]

Hence \( f'(0) = 1 \).

2. A simple calculation of the derivative for \( x \neq 0 \) gives that

\[
f'(x) = \begin{cases} 
  1 + 4x \sin \left( \frac{1}{x} \right) - 2 \cos \left( \frac{1}{x} \right) & x \neq 0 \\
  1 & x = 0.
\end{cases}
\]
III. (10 points) Let $x \in \mathbb{R}$ be a parameter and let

$$\sum_{n} n3^{-n}(x-2)^n$$

be an infinite series.

1. For which values of $x$ does the above series converge absolutely?

2. For which values does it diverges?

Solutions:

1. Using the root test,

$$\lim sup \left| \frac{(n+1)3^{-(n+1)}(x-2)^{n+1}}{n3^{-n}(x-2)^n} \right| = \lim sup \left| \frac{(n+1)(x-2)}{3n} \right| = \frac{|x-2|}{3}$$

Hence, if $|x-2|/3 < 1$ then the infinite series converges absolutely.

For $|x-2| = 3$ is the inconclusive case and we have to check it for these values. But $|x-2| = 3$ implies that $x = 5$ or $x = -1$.

If we plug the value $x = 5$ in the infinite series, we get that

$$\sum_{n} n3^{-n}(x-2)^n = \sum_{n} n3^{-n}(3)^n = \sum_{n} n,$$

which diverges because $n \to \infty$.

If we plug the value $x = -1$ in the infinite series, we get that

$$\sum_{n} n3^{-n}(x-2)^n = \sum_{n} n3^{-n}(-3)^n = \sum_{n} (-1)^n n,$$

which diverges because $(-1)^n n$ does not converge to 0.

2. If $|x-2| > 3$ it diverges.
IV. (10 points) For $n \in \mathbb{N}$, let

$$f_n(x) = n \ x^n(1 - x) \quad x \in [0, 1].$$

1. Find $f(x) = \lim_{n \to \infty} f_n(x)$.

2. Prove that the convergence is not uniform on $[0, 1]$.

3. Does $\lim_{n \to \infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$?

Solutions:

1. It is easy to see that $f_n(0) = 0$ and $f_n(1) = 0$.

   If $x \in (0, 1)$, then $\ln x < 0$ and

   $$\lim_{n \to \infty} n \ x^n(1 - x) = \lim_{n \to \infty} (1 - x)n \exp(n \ln x) = 0$$

   since the exponential converges faster than any polynomial.

   We deduce that $\lim_{n \to \infty} f_n(x) = 0 = f(x)$

2. Let us look for the critical points by computing the derivative of $f_n(x)$.

   $$f'_n(x) = n^2 \ x^{n-1}(1 - x) - nx^n = nx^{n-1}(1 - x) - nx^{n-1}(n(1 - x) - x) = nx^{n-1}(n - x(n + 1)).$$

   Now, $f'_n(x) = 0$ iff $x = 0$ or $x = \frac{n}{n+1}$.

   But $f_n(0) = 0$ while $f_n \left( \frac{n}{n+1} \right) = \frac{n^{n+1}}{(n+1)^2}$. Hence, $\lim f_n \left( \frac{n}{n+1} \right) = \infty$. Which implies that $f_n$ does not converge uniformly on $[0, 1]$.

3. $$\int_0^1 f(x)dx = \int_0^1 0dx = 0.$$

   And

   $$\int_0^1 f_n(x)dx = \int_0^1 n \ x^n(1 - x)dx$$

   $$= \lim_{n \to \infty} \int_0^1 \left( n \ x^n - n \ x^{n+1} \right)dx$$

   $$= \left. \frac{n}{n+1} x^{n+1} - \frac{n}{n+2} x^{n+2} \right|_0^1$$

   $$= \frac{n}{n+1} - \frac{n}{n+2} = \frac{n}{(n+1)(n+2)}.$$
Hence,
\[
\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{n}{(n + 1)(n + 2)} = 0.
\]
Which implies that
\[
\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = \int_0^1 f(x) \, dx = 0.
\]