Partial credit will be awarded for your answers, so it is to your advantage to explain your reasoning and what theorems you are using when you write your solutions. Please answer the questions in the space provided and show your computations.

Good luck!
A. (20pts) Find the general solution of the given system of equations

\[ x' = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} x + \begin{pmatrix} e^t \\ \frac{e^t}{\sqrt{3}e^{-t}} \end{pmatrix} \]
B. (20pts) Let define the following nonlinear system

\[
\begin{align*}
\frac{dx}{dt} &= 1 - y \\
\frac{dy}{dt} &= x^2 - y^2.
\end{align*}
\]

1. Determine all the critical points of the given system of equations.

2. Find the corresponding linear system near each critical point.

3. Find the eigenvalues of each linear system. What conclusions can you then draw about the nonlinear system.

4. Determine all the trajectories (solutions) of the given system and in particular the trajectory that passes through the point (0.0).
C. (10pts) Let us be given the direction field of a predator-prey system (attached) of the following system

\[
\begin{align*}
\frac{dx}{dt} &= x(-1 - 0.3y + 2.5x - x^2) \\
\frac{dy}{dt} &= y(x - 1.5).
\end{align*}
\]

1. Find graphically the critical points.
2. Classify them (graphically).
D. (Bonus question) (10pts) Let us be given the system of nonlinear equations for the damped pendulum

\[
\begin{align*}
\frac{dx}{dt} &= y \\
\frac{dy}{dt} &= -y - \sin x.
\end{align*}
\]

1. Show that the origin is a critical point.

2. Show that while \( V(x, y) = x^2 + y^2 \) is positive definite, \( \dot{V}(x, y) \) takes both positive and negative values in any domain containing the origin, so that \( V \) is not a Liapunov function.
   **Hint:** \( x - \sin x > 0 \) for \( x > 0 \) and \( x - \sin x < 0 \) for \( x < 0 \). Consider these cases with \( y \) positive but \( y \) so small that \( y^2 \) can be ignored compared to \( y \).

3. Using the energy function \( V(x, y) = \frac{1}{2}y^2 + (1 - \cos x) \), show that the origin is a stable critical point. Since there is damping in the system, we can expect that the origin is asymptotically stable. However, it is not possible to draw this conclusion using this Lyapunov function.

4. To show asymptotic stability, it is necessary to construct a better Lyapunov function than the one used in (3). Show that \( V(x, y) = \frac{1}{2}(x^2 + y^2) + x^2 + \frac{1}{2}y^2 \) is such a Lyapunov function.
   **Hint:** From Taylor’s formula with remainder it follows that \( \sin x = x - \alpha x^3/3! \), where \( \alpha \) depends on \( x \) but \( 0 < \alpha < 1 \) for \( -\pi/2 < x < \pi/2 \). Then letting \( x = r \cos \theta, \; x = r \sin \theta \), show that \( \dot{V}(r \cos \theta, r \sin \theta) = -r^2[1 + h(r, \theta)] \), where \( |h(r, \theta)| < 1 \) if \( r \) is sufficiently small.