1. Let \( f(x) = x^{1/3} \) for \( x \in \mathbb{R} \).
   (i) Prove that \( f'(x) = \frac{1}{3}x^{-2/3} \) for \( x \neq 0 \).
   (ii) Show that \( f \) is not differentiable at \( x = 0 \).

2. In each of the following cases, determine the intervals in which the function \( f \) is increasing and find the maxima and minima (if any) in the set where each function is defined
   a) \( f(x) = x^2 - 3x + 4, \quad x \in [0, 2] \).
   b) \( f(x) = \ln(x^2 - 9), \quad |x| > 3 \).
   c) \( f(x) = \frac{\sin x}{x} \) if \( x \neq 0, \ f(0) = 1, \ 0 \leq x \leq \pi/2 \).

3. Define \( f, g \) as follows
   \( f(0) = g(0) = 0 \) and if \( x \neq 0, \ f(x) = \sin(1/x) \) and \( g(x) = x \sin(1/x) \).
   Show that
   a) \( f'(x) = -1/x^2 \cos(1/x) \), if \( x \neq 0; \ f'(0) \) does not exist.
   b) \( g'(x) = \sin(1/x) - 1/x \cos(1/x) \), if \( x \neq 0; \ g'(0) \) does not exist.

4. Suppose that \( f \) is differentiable on \( \mathbb{R} \) and that \( f(0) = 0, \ f(1) = 2, \) and \( f(2) = 2 \).
   a) Show that there exists \( c_1 \in (0, 1) \) such that \( f'(c_1) = 2 \).
   b) Show that there exists \( c_2 \in (1, 2) \) such that \( f'(c_2) = 0 \).
   c) Show that there exists \( c_3 \in (0, 2) \) such that \( f'(c_3) = 1 \).

5. Use the mean value theorem to establish the following inequalities
   a) \( \sin x \leq x, \) for \( x \geq 0 \).
   b) \( \left| \frac{\sin(ax) - \sin(bx)}{x} \right| \leq |a - b|, \) for \( x \neq 0 \).

6. Show that if \( \sum |a_n - a_{n+1}| < \infty \) then \( (a_n) \) converges, but not conversely.

7. Let \( (a_n)_n \) and \( (b_n)_n \) be sequences in \( \mathbb{R} \).
   Show that if \( \sum b_n \) converges and \( \sum |a_n - a_{n+1}| < \infty \), then \( \sum a_n b_n \) converges.

8. Let \( (x_n)_n \) be a sequence of real numbers and let \( y_n = x_n - x_{n+1} \) for each \( n \in \mathbb{N} \).
   (a) Prove that the series \( \sum_{n=1}^{\infty} y_n \) converges if and only if the sequence \( (x_n)_n \) converges.
   (b) If \( \sum_{n=1}^{\infty} y_n \) converges, what is the sum?

9. Determine whether each series converges or diverges. Justify your answer.
   (a) \( \sum \frac{2^n}{n} \)
   (b) \( \sum \frac{\sin^2 n}{n^2} \)
   (c) \( \sum \frac{1}{n \sqrt{n+1}} \)
   (d) \( \sum \frac{(-1)^n \log n}{n} \)
10. Suppose \( a > 0 \). Let \( x_1 = \sqrt{a} \), and define \( x_{n+1} = \sqrt{a + x_n} \) for \( n \geq 1 \).

(a) Show that \( x_n < 1 + \sqrt{a} \) for all \( n \).
(b) Show that \( (x_n)_{n \geq 1} \) is an increasing sequence.
(c) Show that \( (x_n)_{n \geq 1} \) converges and find its limit.

11. Let \( f_n(x) = \frac{x}{x+n} \) for \( x \geq 0 \).

(a) Show that \( f(x) = \lim f_n(x) = 0 \) for all \( x \geq 0 \).
(b) Show that if \( t > 0 \), the convergence is uniform on \([0, t]\).
(c) Show that the convergence is not uniform \([0, \infty)\).

12. Let \( f_n(x) = \frac{nx}{1+nx} \) for \( x \geq 0 \).

(a) Find \( f(x) = \lim f_n(x) = 0 \).
(b) Show that if \( t > 0 \), the convergence is uniform on \([t, \infty)\).
(c) Show that the convergence is not uniform \([0, \infty)\).

13. Let \( f_n(x) = \frac{nx}{e^x} \) for \( x \in [0, 2] \).

(a) Show that \( \lim f_n(x) = 0 \) for all \( x \in [0, 2] \).
(b) Show that the convergence is not uniform on \([0, 2]\).
(c) Let \( 0 < t < 2 \). Determine on which interval, \([0, t]\) or \([t, 2]\), the convergence is uniform. Justify your answer.

14. Let \( f_n(x) = nx^n(1-x) \) for \( x \in [0, 1]\).

(a) Find \( f(x) = \lim f_n(x) = 0 \).
(b) Show that the convergence is not uniform \([0, 1]\).
(c) Does \( \lim \int_0^1 f_n(x)dx = \int_0^1 f(x)dx \)?

15. Let \( f_n(x) = x + \frac{1}{n} \) and \( f(x) = x \) for \( x \in \mathbb{R} \).

(a) Show that \( (f_n) \) converges uniformly to \( f \) on \( \mathbb{R} \).
(b) Show that \( (f_n') \) converges pointwise to \( f \) on \( \mathbb{R} \) but not uniformly.


(a) \( \sum \frac{x^{2n}}{(n+x)^2} \), \([0, 1]\)
(b) \( \sum \frac{x^2}{n^2} \), \([5, \infty)\)
(c) \( \sum \frac{n^{-x}}{n} \), \((\sqrt{2}, \infty)\)
(d) \( \sum \frac{1}{n} \sqrt{\frac{\sin nx}{n}} \), \(x \in \mathbb{R}\)
(e) \( \sum \frac{1}{1+(nx)^2} \), \((0, 1]\)