Coarse-graining as a numerical approximation scheme
   Error estimates in CG
   Adaptive coarse-graining
   Microscopic Reconstruction – Reversing the CG map

A statistics approach to Coarse-graining
   Parametrization and Model Selection Challenges
Hierarchical Coarse-graining in macromolecules

Coarse-graining of polymers; proteins; biomembranes

CG procedure, "super-atoms":

[Diagram of a polymer chain showing coarse-grained segments]
CG of Macromolecules

**Microscopics: United Atom (UA) Model**

- Continuum model: \( X \in (\mathbb{R}^3)^N \) – positions of \( n \) atoms on one macromolecule; \( m \) macromolecules; \( N = nm \).
- Hamiltonian \( H_N(X) = H_b(X) + H_{nb}(X) + H_{wall} + H_{kin} \)

**Bonded Interactions**: Gaussian, FENE, etc. short-range

\[
H_b(X) = \sum_i U_b(\theta_i, \phi_i, r_i) \quad \text{short-range}
\]

**Non-Bonded Interactions**: 12-6 Lennard Jones long-range

\[
H_{nb}(X) = \sum_{i,j} U_{nb}^{LJ}(|x_i - x_j|)
\]
**Example:** Dimers

\[
H_N(X) = \sum_{i=1}^{N} U_b(|x_{2i} - x_{2i+1}|) + \sum_{i=1}^{2N} \sum_{j \neq i} U_{nb}^{LJ}(|x_i - x_j|)
\]

**Equilibrium Gibbs measure** at \( \beta = \frac{1}{kT} \).

\[
\mu_{N,\beta}(dX) = \frac{1}{Z_{N,\beta}} e^{-\beta H_N(X)} P_N(dX)
\]

- **Prior:** \( P_N(dX) = \prod_{i \in \Lambda} \rho(dX_i) = \prod_{i \in \Lambda} dX_i \).
- **Partition function:** \( Z_{N,\beta} \)
- \( T = \infty \) vs. \( T = 0 \) regimes.
- **Extensivity:** \( H_N \sim O(N) \).
Molecular Dynamics at fixed temperature via ”Langevin thermostat”:

Microscopic dynamics

\[ \dot{q} = \nabla_p H(p, q) \]
\[ \dot{p} = -\nabla_q H(p, q) - \gamma p + \sqrt{2/\beta} \dot{W} \]

CG map: \((P, Q) = \Pi(p, q)\)

Effective equations of motion

Coarse-grained Hamiltonian \(\tilde{H}(P, Q)\)

\[ \dot{Q} = \nabla_P \tilde{H}(P, Q) \]
\[ \dot{P} = -\nabla_Q \tilde{H}(P, Q) - \tilde{\gamma} P + \sigma \dot{W} \]

\(\gamma\): damping due to friction

\(W\): multi-dimensional Brownian motion with covariance matrix coupling \(\beta\) and \(\gamma\) through fluctuation-dissipation relation; Gibbs measure is invariant.

- ad hoc CG: wrong predictions of diffusion, crystallization, phase transitions

- without numerical analysis no indication of wrong phenomenon being deduced from simulation.

- adaptive change of CG difficult
Some of the references in CG - soft matter literature

Doi et. al. *J.Chem.Phys.* '02;
Kremer et. al. *Macromolecules* '06;
Müller-Plathe *Chem.Phys.Chem* '00;
Laaksonen et. al. *Soft Matter* '03;
Deserno et. al. *Nature* '07;
Shell *J.Chem.Phys.*'08, '12, '13;
...
Renormalization-Exact CG Hamiltonian $\tilde{H}(Q)$


Key objects for equilibrium CG:

1. **Coarse-graining map:**

   $$\Pi X = Q = (Q_1, \ldots, Q_m) \in Q$$

   where $Q_i \in \mathbb{R}^3$ (or $\mathbb{R}^6$ if velocities are included)

2. **Renormalization map** $\mapsto$ CG Hamiltonian (Why this quantity?)

   $$\tilde{H}(Q) = -\frac{1}{\beta} \log \int_{\{X|\Pi X = Q\}} e^{-\beta H_N(X)} \, dX$$
Renormalization-Exact CG Gibbs $\bar{\mu}_M,\beta$

1. **Coarse observables:** $f(Q) = f(\Pi X)$, $X = (\Pi X, \Pi X^\perp) = (Q, Q^\perp)$
2. **Exactly CG Gibbs measure** $\bar{\mu}_M,\beta$: Defined as $\bar{\mu}_M,\beta = \mu_{N,\beta} \circ \Pi^{-1}$, i.e.

$$\int f(Q) \bar{\mu}_M,\beta(Q) dQ = \int f(\Pi X) \mu_{N,\beta}(X) dX .$$

- Then, an easy calculation on the observables $f = f(Q)$ shows:

$$\int f(Q) \bar{\mu}_M,\beta(Q) dQ = \int f(\Pi X) \mu_{N,\beta}(X) dX = \int f(Q) \left[ \int_{\{X|\Pi X = Q\}} \mu_{N,\beta}(X) dX \right] dQ$$

- We replace $\mu_{N,\beta}(X) \sim e^{-\beta H_N(X)}$ in $[...]$ and can now define

$$\bar{H}(Q) = -\frac{1}{\beta} \log \int_{\{X|\Pi X = Q\}} e^{-\beta H_N(X)} dX$$

- We obtain the exactly CG Gibbs measure (take partition functions into account!):

$$\bar{\mu}_M,\beta(dQ) = \frac{1}{Z_M} e^{-\beta \bar{H}_M(Q)} dQ .$$
Approximating the exact CG Gibbs $\tilde{\mu}_{M, \beta}$

**Break-up of computational task:**
**Simplifying assumptions-approximations**
(i) $\bar{H}$ decouples:

$$\bar{H}(Q) \approx \bar{H}_b + \bar{H}_{nb} \approx \sum_{\text{CG var.}} \bar{U}_b + \bar{U}_{nb}$$

where

$$\bar{H}_b(Q) = -\frac{1}{\beta} \log \int_{\{X | \Pi X = Q\}} e^{-\beta H_b(X)} dX$$

and

$$\bar{H}(Q) = -\frac{1}{\beta} \log \int_{\{X | \Pi X = Q\}} e^{-\beta [H_b(X) + H_{nb}(X)]} dX$$

(ii) $\bar{U}_b = \bar{U}_b^\theta + \bar{U}_b^\phi + \bar{U}_b^r$ where each term depends *only* on torsion angle $\phi$, rotation angle $\theta$ and distance $r$ respectively between successive CG particles.

(iii) We construct $\bar{U}_{nb}$ next:
How to calculate the CG non-bonded interactions $\bar{U}_{nb}$: McCoy-Curro scheme, *Macromolecules* ’98.

For two isolated small molecules with centers of mass at $q_1$, $q_2$:

$$U_{nb}(|q_1 - q_2|) = -\frac{1}{\beta} \log \int_{\{X | \Pi X = (q_1, q_2)\}} e^{-\beta H(X)} \, dX$$

- The calculation is computationally feasible but disregards multi-body interactions.
Challenges in coarse-graining methods

Often: wrong predictions in dynamics, phase transitions, melt structure, crystallization, etc. See for instance:

- CG in polymers: sensitive dependence to temperature

low vs. high temperature

Doi et al. *J. Chem. Phys.* '02

- "classical" example: 1-D nearest neighbor Ising vs. Curie-Weiss (see Talk 2)
Mathematics and Numerics of CG

Approximating CG Hamiltonian $\tilde{H}(Q)$

**Step 1:** identify a base state $\tilde{H}^{(0)}_M(Q)$ and a “small” parameter $\epsilon$

$$\tilde{H}(Q) = \tilde{H}^{(0)}_M(Q) - \frac{1}{\beta} \log \mathbb{E}[e^{-\beta(H_N-\tilde{H}^{(0)}_M)} | Q].$$

**Step 2:** choose a suitable expansion and expand the approximation error in $\epsilon$

$$\tilde{H}_M(Q) = \tilde{H}^{(0)}_M(Q) + \tilde{H}^{(1)}_M(Q) + \tilde{H}^{(2)}_M(Q) + \cdots + \tilde{H}^{(p)}_M(Q) + N \times O(\epsilon^p),$$

**Heuristics:** Expansion of $e^{\Delta H}$ and log:

$$= \mathbb{E} [\Delta H | Q] + \mathbb{E} [(\Delta H)^2 | Q] - \mathbb{E} [\Delta H | Q]^2 + O((\Delta H)^3)$$

however...

$$\Delta H \equiv H_N - \tilde{H}^{(0)}_M = N \cdot O(\epsilon)$$

- The role of the complementary "troublemakers": Fluctuations and Extensivity
Suitable choice for $\bar{H}_M^{(0)}$? McCoy-Curro scheme? What is the error? Can we ”do better”?

**Step 3:** The zero-th order approximation $\bar{H}^{(0)}$ should satisfy

$$\mathbb{E} \left[ H_N - \bar{H}_m^{(0)} \mid Q \right] = 0.$$

- **Rigorous analysis – Cluster expansion:** around $\bar{H}_m^{(0)}$.

*Cluster expansions* developed in statistical physics for controlling measures on high-dimensional spaces.

K., Plechac, Rey-Bellet, Tsagkarogiannis, [M^3 AN, ’07]
1. **Relative Entropy:** Bounds on the “distance” between the exact $\mu^0_{M,\beta}(dQ)$ and the approximate $\mu_{M,\beta}(dQ)$.

**Error control in terms of relative entropy estimates:** Information loss in CG

$$R(\pi_1 \mid \pi_2) = \frac{1}{N} \int_S \log \frac{d\pi_1}{d\pi_2} \pi_1(d\sigma).$$

2. **Observables:** Bounds on the weak error:

$$\mathbb{E}_{\pi_X_0}[f(\Pi X)] - \mathbb{E}_{Q_0}[f(Q)]$$
Relative Entropy (Kullback-Leibler Divergence)

- **Pseudo-distance** between probability measures $P$, $Q$:

$$
\mathcal{R} (P \mid Q) := \int \log \left( \frac{dP}{dQ} \right) dP
$$

- **Properties**: (i) $\mathcal{R} (P \mid Q) \geq 0$ and
  (ii) $\mathcal{R} (P \mid Q) = 0$ iff $P = Q$ a.e.

- **Observables**: Csiszar-Kullback-Pinsker inequality:

$$
|\mathbb{E}_P [\phi] - \mathbb{E}_Q [\phi]| \leq \|\phi\|_\infty \sqrt{2 \mathcal{R} (P \mid Q)}
$$

*(sharper bounds also available—see Talk 6)*

**Coarse-Graining–Error Quantification and Parameterization using RE:**


Coarse-Graining - Two classes of molecular models

1. Coarse-graining of macromolecules

2. Stochastic lattice dynamics/ KMC
Coarse Graining in Lattice Systems

Divide lattice of size $N$ into $M$ cells with $q$-particles in each cell.
Coarse Graining in Lattice Systems

Divide lattice of size $N$ into $M$ cells with $q$-particles in each cell

**Coarse map:**

\[ \Pi : \Sigma_N \rightarrow \Sigma_M \]

\[ \sigma \mapsto \eta := \{ \eta(k) = \sum_{x \in C_k} \sigma(x) \} \]
Coarse Graining in Lattice Systems

Divide lattice of size $N$ into $M$ cells with $q$-particles in each cell

Coarse map:

$$\Pi : \Sigma_N \rightarrow \Sigma_M$$

$$\sigma \mapsto \eta := \{ \eta(k) = \sum_{x \in C_k} \sigma(x) \}$$

Renormalization Group map:

$$\bar{H}(\eta) = -\frac{1}{\beta} \log \int \exp\{-\beta H_N(\sigma)\} P(d\sigma | \eta)$$
1-D example: short-range interactions

Approximation of RG map $\bar{H}(\eta)$ by a computable $\bar{H}^0(\eta)$:

$$H_N(\sigma) = \sum_k H_k(\sigma) + \sum_k W_{k,k+1}(\sigma)$$

$H_k$: energy for the cell $C_k$ with free boundary conditions
$W_{k,k+1}$: short-range interactions between cell $k$ and cell $k+1$.

$$e^{-\beta H_N} P_N(d\sigma | \eta) = \prod_{k: \text{odd}} \left[ e^{-\beta (W_{k-1,k} + W_{k,k+1})} e^{-\beta H_k} P_k(d\sigma^k | \eta(k)) \right] \times \prod_{k: \text{even}} e^{-\beta H_k} P_k(d\sigma^k | \eta(k))$$

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A simple example

-When $W_{k,k+1}$ are disregarded (e.g. high temps $\beta \ll 1$), there are intra-cell interactions, but there are no CG cell correlations:

$$
\bar{H}_m^{(0)}(\eta) = \sum_k \bar{U}_k^{(0)}(\eta_k) = -\sum_k \frac{1}{\beta} \log \int e^{-\beta H_k(\sigma)} P_k(d\sigma^k|\eta(k))
$$

**Sampling over a single coarse cell with free boundary conditions**

A simple example

-When $W_{k,k+1}$ are disregarded (e.g. high temps $\beta \ll 1$), there are intra-cell interactions, but there are no CG cell correlations:

$$
\bar{H}_m^{(0)}(\eta) = \sum_k \bar{U}_k^{(0)}(\eta_k) = - \sum_k \frac{1}{\beta} \log \int e^{-\beta H_k(\sigma)} P_k(d\sigma^k|\eta(k))
$$

Sampling over a single coarse cell with free boundary conditions

Spatial correlations between CG cells

Double/Triple terms in CG short range interactions: revisit earlier approximation:

\[
e^{-\beta H^s_N} \mathcal{P}_N(d\sigma | \eta) = \prod_{k: \text{odd}} \left[ e^{-\beta(W_{k-1,k}+W_{k,k+1})} e^{-\beta H^s_k} \mathcal{P}_k(d\sigma^k | \eta(k)) \right] \times \prod_{k: \text{even}} e^{-\beta H_k} \mathcal{P}_k(d\sigma^k | \eta(k))
\]

Triple body term lurks inside: \[\prod_{k: \text{odd}} [...]\]

Rigorous Analysis: K., Plechac, Rey-Bellet, Tsagkarogiannis ESAIM M2AN '07, SIAM Sci. Comp.' 09, Math. Comp. '14
Rigorous results - Long-range interactions

K., Plechac, Rey-Bellet, Tsagkarogiannis, *ESAIM: M3AN*, ’07

- **CG Hamiltonian–Renormalization Group Map**: \( N = mq \)

\[
e^{-\beta \tilde{H}_m(\eta)} = \int e^{-\beta H_N(\sigma)} P_N(d\sigma \mid \eta) \equiv \mathbb{E}[e^{-\beta H_N} \mid \eta]
\]

- **Correction terms** around a first "good guess" \( \tilde{H}_m^{(0)} \):

\[
\tilde{H}_m(\eta) = \tilde{H}_m^{(0)}(\eta) - \frac{1}{\beta} \log \mathbb{E}[e^{-\beta (H_N - \tilde{H}_m^{(0)})} \mid \eta], \quad m = N, N - 1, ...
\]

- **Heuristics**: Expansion of \( e^{\Delta H} \) and log:

\[
= \mathbb{E} [\Delta H \mid \eta] + \mathbb{E} [(\Delta H)^2 \mid \eta] - \mathbb{E} [\Delta H \mid \eta]^2 + \mathcal{O}((\Delta H)^3)
\]

formal calculations inadequate since:

\[
\Delta H \equiv H_N - \tilde{H}_m^{(0)} = N \cdot \mathcal{O}(\epsilon)
\]
Can we use ”independence” in order to avoid the dependence of the error on the system size $N$?

- Assume for a moment that:

$$H_N - \bar{H}_m^{(0)} = \sum_k \Delta_k(\sigma^k)$$

where $\sigma^k$ is the configuration restricted only on the coarse cell $k$:

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- Then, in

$$\bar{H}_m(\eta) = \bar{H}_m^{(0)}(\eta) - \frac{1}{\beta} \log \mathbb{E}[e^{-\beta(H_N - \bar{H}_m^{(0)})} | \eta],$$

we have

$$\log \mathbb{E}[e^{-\beta(H_N - \bar{H}_m^{(0)})} | \eta] = \log \mathbb{E}[\prod_k e^{-\beta\Delta_k(\sigma^k)} | \eta] = \sum_k \log \mathbb{E}[e^{-\beta\Delta_k(\sigma^k)} | \eta]$$

- In this case we can carry out standard numerical analysis approximations for each $k$-term separately.

- Can that be done rigorously?
More heuristics

Can we use “independence” in order to avoid the dependence of the error on the system size $N$?

- Assume for a moment that:

$$H_N - \bar{H}_m^{(0)} = \sum_k \Delta_k(\sigma^k)$$

where $\sigma^k$ is the configuration restricted only on the coarse cell $k$:

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- Then, in

$$\bar{H}_m(\eta) = \bar{H}_m^{(0)}(\eta) - \frac{1}{\beta} \log \mathbb{E}[e^{-\beta(H_N - \bar{H}_m^{(0)})} | \eta],$$

we have

$$\log \mathbb{E}[e^{-\beta(H_N - \bar{H}_m^{(0)})} | \eta] = \log \mathbb{E}[\prod_k e^{-\beta \Delta_k(\sigma^k)} | \eta] = \sum_k \log \mathbb{E}[e^{-\beta \Delta_k(\sigma^k)} | \eta]$$

- In this case we can carry out standard numerical analysis approximations for each $k$-term separately.
- Can that be done rigorously?

But first we will need to build a good $\bar{H}_m^{(0)}(\eta)$.
Building an approximate CG Hamiltonian $\tilde{H}^{(0)}(\eta)$

$$\tilde{H}^{(0)}(\eta) = -\frac{1}{2} \sum_{l \in \tilde{\Lambda}_M} \sum_{k \neq l} \tilde{J}(k, l) \eta(k) \eta(l) - \frac{1}{2} \tilde{J}(0, 0) \sum_{l \in \tilde{\Lambda}_M} \eta(l)(\eta(l) - 1) + \sum_{l \in \tilde{\Lambda}_M} \tilde{h}(l) \eta(l)$$

- $\mathbb{E}\left[ H_N - \tilde{H}^{(0)} | \eta \right] = 0$ (reason for being 2nd order scheme-see earlier). Involves two-body CG interaction only:

$$\tilde{J}(k, l) \eta(k) \eta(l) = \int \sum_{x \in C_k, y \in C_l} J(x - y) \sigma(x) \sigma(y) P_N(d\sigma | \eta_k, \eta_l)$$

Where

$$\tilde{J}(k, l) = \frac{1}{q^2} \sum_{x \in C_k} \sum_{y \in C_l, y \neq x} J(x - y)$$

- Approximating Gibbs measure:

$$\tilde{\mu}^{(0)}_{M, q, \beta}(d\eta) = \frac{1}{Z_{M, q, \beta}} e^{-\beta \tilde{H}^{(0)}(\eta)} P_q(d\eta),$$

Markos Katsoulakis, University of Massachusetts, Amherst

Talk 3: Coarse-graining of high-dimensional stochastic models
Rigorous analysis – Cluster expansions

**Idea:** Identify clusters that do not "communicate" e.g. $\prod_k$ term in the simple example earlier. Then factorize and Taylor-expand as in the heuristic argument.

**Step 1:** Rewrite

$$
\mathbb{E}
\left[
\exp\left(-\beta(H_N - \bar{H}^{(0)})\right) \mid \eta
\right] = \int \prod_{k \leq l} (1 + (e^{-\beta\Delta_{kl}J(\sigma)} - 1)) \, P_N(d\sigma \mid \eta)
$$

where

$$
\Delta_{kl}J(\sigma) = \frac{1}{2} \sum_{x \in C_k} \sum_{y \in C_l} (J(x - y) - \bar{J}(k, l))\sigma(x)\sigma(y)
$$

**Step 2:** Assume $e^{-\beta \Delta_{kl}J(\sigma)} - 1$ small and expand

$$
\prod_{k \leq l} (1 + (e^{-\beta\Delta_{kl}J(\sigma)} - 1)) = 1 + \sum_{G \in \mathcal{G}_M} \prod_{\{k, l\} \in G} (e^{-\beta\Delta_{kl}J(\sigma)} - 1))
$$

Convergence criterion for the resulting series (Kotecký-Preiss-Dobrushin)
Corrections to the Hamiltonian $\tilde{H}^{(0)} \mapsto \textbf{Multi-body terms}$

$$\tilde{H}_m(\eta) = \tilde{H}_m^{(0)}(\eta) + \tilde{H}_m^{(1)}(\eta) + \ldots$$

$$\tilde{H}_m^{(1)}(\eta) = \beta \sum_{k_1} \sum_{k_2 > k_1} \sum_{k_3 > k_2} j^2_{k_1 k_2 k_3} (-E_1(k_1)E_2(k_2)E_1(k_3) + \ldots$$

- $E_r(k) \equiv E_r(\eta(k)) = (2\eta(k)/q - 1)^r + o_q(1)$
- “Moments” of interaction potential $J$:

$$j^2_{k_1 k_2 k_3} = \sum_{x \in C_{k_1}} \sum_{y \in C_{k_2}} \sum_{z \in C_{k_3}} (J(x-y) - \bar{J}(k_1, k_2))(J(y-z) - \bar{J}(k_2, k_3))$$

-Computational complexity of the corrections?
-Compression of $\tilde{H}^{(1)}$?

Are, K. Plechac, Rey-Bellet, SIAM J. Sci. Comp. ’09
Error Quantification in CG Schemes

**Theorem 1**: (A priori error analysis)
Define the new parameter
\[ \epsilon \equiv \beta \frac{q}{L} \| \nabla J \|_1 \]
where \( |C_k| = q \) is the scale of CG, \( L \) is the interaction range.

1. **Approximation of the CG free-energy landscape**:
\[ \tilde{H}_m(\eta) = H_m^{(0)}(\eta) - \frac{1}{\beta} \log \mathbb{E}[e^{-\beta(H_N - \tilde{H}_m^{(0)})} | \eta] = H_m^{(0)}(\eta) + \tilde{H}_m^{(1)}(\eta) + NO(\epsilon^3). \]

2. **Loss of information during coarse-graining**
   - For the specific relative entropy: \( \mathcal{R}(\mu | \nu) := \frac{1}{N} \sum_\sigma \log \left\{ \frac{\mu(\sigma)}{\nu(\sigma)} \right\} \mu(\sigma) \)
   \[ \mathcal{R}(\tilde{\mu}_M,q,\beta | \mu_{N,\beta} \circ \Pi^{-1}) = O(\epsilon^{\alpha+2}) \cdot \diamond \]

   - \( \Pi \sigma = \text{Projection} \) on coarse variables = \( \sum_{y \in C_k} \sigma(y) \).

K., Plechac, Rey-Bellet, Tsagkarogiannis, [ESAIM: M3AN, ’07]
Multi-body terms in CG Hamiltonians

1. Cluster expansion CG Hamiltonian:  \( \bar{H}_m(\eta) = \bar{H}^{(0)}_m(\eta) + \bar{H}^{(1)}_m(\eta) \)

   Multi-body terms:

   \[
   \bar{H}^{(1)}(\eta) = \beta \sum_{k_1} \sum_{k_2 > k_1} \sum_{k_3 > k_2} j_{k_1 k_2 k_3} (-E_1(k_1)E_2(k_2)E_1(k_3) + ...)
   \]

Typically omitted, but essential to capture phase transitions, switching times:


Some computational tests

**CG Arrhenius lattice dynamics** Hysteresis diagram

1. Piecewise constant potentials of radius $L$:

![Hysteresis diagram](image)

Magnetization: interaction $L=8$, $J_0=1$, $\beta=2$ (mean field: $\beta_c = 1$)

- $q=1$
- $q=8$
- $q=8$ corr
- Curie-Weiss
2. **Switching Time PDFs/Autocorrelations-corrections**

- Probability density function (pdf) of the exit time

- $q = 1$, MC
- $q = 100$, 3rd-order CGMC
- $q = 50$, 2nd-order CGMC
- $q = 100$, 2nd-order CGMC

- $q = 100$, 3rd-order CGMC
- $q = 100$, with corrections

- Microscopic process

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Are, K. Plechac, Rey-Bellet, SIAM J. Sci. Comp. '09
Recent related rigorous analysis for CG:

- **Off lattice systems:**

- **Short and long range interactions:**

- **Multi-level CG methods:**

- **Non-equilibrium:**
  K. Plechac, *J. Chem. Phys.* '13; see Talk 4
Adaptive CG and a posteriori error

[K., Plechac, Rey-Bellet, Tsagkarogiannis, J.Non-Newt. Fluid Mech., '08]

1. Cluster expansions → a posteriori expansion for the relative entropy.

\[
\bar{H}_m(\eta) = \bar{H}_m^{(0)}(\eta) - \frac{1}{\beta} \log \mathbb{E}[e^{-\beta(H_N - \bar{H}_m^{(0)})} | \eta] = \bar{H}_m^{(0)}(\eta) + \bar{H}_m^{(1)}(\eta) + \ldots
\]

\[
\bar{H}^{(1)}(\eta) = \beta \sum_{k_1} \sum_{k_2 > k_1} \sum_{k_3 > k_2} j^2_{k_1 k_2 k_3} \left( -E_1(k_1)E_2(k_2)E_1(k_3) + \ldots \right)
\]

- \( E_r(k) \equiv E_r(\eta(k)) = (2\eta(k)/q - 1)^r + o_q(1) \)
- “Moments” of interaction potential \( J \):

\[
j^2_{k_1 k_2 k_3} = \sum_{x \in C_{k_1}} \sum_{y \in C_{k_2}} \sum_{z \in C_{k_3}} (J(x - y) - \bar{J}(k_1, k_2))(J(y - z) - \bar{J}(k_2, k_3))
\]

The error indicator \( R(.) \) is given by the terms \( \bar{H}^{(1)}, \bar{H}^{(2)} \) and depends only on the coarse variable \( \eta \):

\[
\mathcal{R} \left( \mu_{m,q}^{(0)} | \mu_N \circ \Pi \right) = \mathbb{E}[R(\eta)] + \log \left( \mathbb{E}[e^{-R(\eta)}] \right) + O(\epsilon^3)
\]

or the error indicator

\[
\text{err} \sim \mathbb{E} \left[ \frac{R(\eta) - ER(\eta)}{\sqrt{N}} \right]^2
\]
A mathematical prototype:

*Competing short and long range potentials*

\[
H_N = -K \sum_{|x-y|=1} \sigma(x)\sigma(y) - \frac{J}{2N} \sum_{x,y} \sigma(x)\sigma(y) + h \sum_x \sigma(x)
\]

Exact solution in 1D/2D (M. Kardar, PRB ’83)
Benchmarking with a "worst-case" scenario:

N.N. antiferromagnetic system:

- pattern $\sim$ lattice size – local averages are not suitable CG variables.

Adaptive computation of phase diagram of anti-ferromagnetic system

However...

Error estimator $\mapsto$ CGMC refines down to $q=1$ in parts of phase diagram (as it should).
Other work on spatial adaptivity for CG systems
- Chaterjee, K., Vlachos, *Phys. Rev. E* '05; *J. Chem. Phys.* ’05 (also based on a posteriori indicators)
- Clementi, Kremer et al *J. Chem Phys* ’08
- Nielsen, Moore, and Ensing, *Phys. Rev. Lett.* ’11
In polymer science literature: [Tschöp et al Acta Polymer. ’98]

Mathematical formulation:
1. **CG Scheme**: \( \bar{\mu}_{\text{app}}^{M} (d\eta) \approx \bar{\mu}_{\text{M}} (d\eta) \)
2. **Reconstruction**: Construct a “suitable” conditional probability \( \nu_{N} (d\sigma | \eta) \) and define the approximate microscopic measure

\[
\mu_{\text{app}}^{N} (d\sigma) := \nu_{N} (d\sigma | \eta) \bar{\mu}_{\text{app}}^{M} (d\eta).
\]

Efficiency of the reconstruction using relative entropy:

\[
\mathcal{R} (\mu_{\text{app}}^{N} | \mu_{\text{N}}) = \mathcal{R} (\bar{\mu}_{\text{app}}^{M} | \bar{\mu}_{\text{M}}) + \int \mathcal{R} (\nu_{N} (\cdot | \eta) | \mu_{N} (\cdot | \eta)) \bar{\mu}_{\text{app}}^{M} (d\eta)
\]

- Specific relative entropy: \( \mathcal{R} (\mu | \nu) := \frac{1}{N} \sum_{\sigma} \log \left\{ \frac{\mu(\sigma)}{\nu(\sigma)} \right\} \mu(\sigma) \).
Microscopic Reconstruction – Reverse CG map

Mathematical formulation:
1. CG Scheme: \( \bar{\mu}_M^{\text{app}}(d\eta) \approx \bar{\mu}_M(d\eta) \)
2. Reconstruction: Construct a “suitable” conditional probability \( \nu_N(d\sigma|\eta) \) and define the approximate microscopic measure

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\bar{\mu}_N^{\text{app}}(d\sigma) := \nu_N(d\sigma|\eta) \bar{\mu}_M^{\text{app}}(d\eta).
\]

Efficiency of the reconstruction using relative entropy:

\[
\mathcal{R} \left( \mu_N^{\text{app}} | \mu_N \right) = \mathcal{R} \left( \bar{\mu}_M^{\text{app}} | \bar{\mu}_M \right) + \int \mathcal{R} \left( \nu_N(\cdot|\eta) | \mu_N(\cdot|\eta) \right) \bar{\mu}_M^{\text{app}}(d\eta)
\]

- Specific relative entropy: \( \mathcal{R}(\mu|\nu) := \frac{1}{N} \sum_{\sigma} \log \left\{ \frac{\mu(\sigma)}{\nu(\sigma)} \right\} \mu(\sigma). \)

Example: \( \bar{\mu}_M^{\text{app}}(d\eta) = \bar{\mu}_M^{(0)}(d\eta), \; \nu_N(d\sigma|\eta) = P_N(d\sigma|\eta). \)
Coarse-graining as a numerical approximation scheme
A statistics approach to Coarse-graining

Microscopic Reconstruction – Reverse CG map

In polymer science literature: [Tschöp et al Acta Polymer. ’98]

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**Efficiency of the reconstruction** using relative entropy:

\[
\mathcal{R}(\mu_N^{\text{app}}|\mu_N) = \mathcal{R}(\bar{\mu}_M^{\text{app}}|\bar{\mu}_M) + \int \mathcal{R}(\nu_N(\cdot|\eta)|\mu_N(\cdot|\eta)) \bar{\mu}_M^{\text{app}}(d\eta)
\]

- **Specific relative entropy:** \( \mathcal{R}(\mu|\nu) := \frac{1}{N} \sum_{\sigma} \log \left\{ \frac{\mu(\sigma)}{\nu(\sigma)} \right\} \mu(\sigma) \).

**Example:**
- \( \bar{\mu}_M^{\text{app}}(d\eta) = \bar{\mu}_M^{(0)}(d\eta) \), \quad \nu_N(d\sigma|\eta) = P_N(d\sigma|\eta)
  - **a.** \( P_N(\sigma|\eta) \) is a product measure \( \Rightarrow \) "local" reconstruction at each coarse-cell;
Microscopic Reconstruction – Reverse CG map

In polymer science literature: [Tschöp et al Acta Polymer. ’98]

**Mathematical formulation:**
1. **CG Scheme:** \( \tilde{\mu}_M^{\text{app}}(d\eta) \approx \tilde{\mu}_M(d\eta) \)
2. **Reconstruction:** Construct a “suitable” conditional probability \( \nu_N(d\sigma|\eta) \) and define the approximate microscopic measure
   \[
   \mu_N^{\text{app}}(d\sigma) := \nu_N(d\sigma|\eta)\tilde{\mu}_M^{\text{app}}(d\eta).
   \]

**Efficiency of the reconstruction** using relative entropy:

\[
\mathcal{R}(\mu_N^{\text{app}}|\mu_N) = \mathcal{R}(\tilde{\mu}_M^{\text{app}}|\tilde{\mu}_M) + \int \mathcal{R}(\nu_N(\cdot|\eta)|\mu_N(\cdot|\eta)) \tilde{\mu}_M^{\text{app}}(d\eta)
\]

- **Specific relative entropy:** \( \mathcal{R}(\mu|\nu) := \frac{1}{N}\sum_{\sigma} \log \left\{ \frac{\mu(\sigma)}{\nu(\sigma)} \right\} \mu(\sigma) \).

**Example:**
\[
\tilde{\mu}_M^{\text{app}}(d\eta) = \tilde{\mu}_M^{(0)}(d\eta), \quad \nu_N(d\sigma|\eta) = P_N(d\sigma|\eta),
\]
   a. \( P_N(\sigma|\eta) \) is a product measure \( \Rightarrow \) ”local” reconstruction at each coarse-cell;
   b. Numerical error estimates.

Systematic mathematical study: [Trashorras, Tsagkarogiannis SIAM Num. Analysis’ 10]:

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Markos Katsoulakis, University of Massachusetts, Amherst

Talk 3: Coarse-graining of high-dimensional stochastic models
Outline

Coarse-graining as a numerical approximation scheme
  Error estimates in CG
  Adaptive coarse-graining
  Microscopic Reconstruction – Reversing the CG map

A statistics approach to Coarse-graining
  Parametrization and Model Selection Challenges
Coarse-graining as a numerical approximation scheme
A statistics approach to Coarse-graining

A statistics perspective: Inverse Monte Carlo - Equilibrium

- Build parametrized approx. of Gibbs states:

\[ \mu_{\text{app}} \sim e^{-\beta H_{\text{app}}(\eta;\theta)} \]

- Find optimal values of parameters \( \theta^* \), such that

\[ \min_{\theta} \sum_i |E_{\mu}[\phi_i] - E_{\mu_{\text{app}}}[\phi_i]|^2 \]

for selected observables \( \phi_i \).

  e.g. F. Muller-Plathe, *Chem. Phys. Chem.* (2002)

- Parametrization depends on specific observable(s) \( \phi \).

- Special case: Force-matching methods,

  [G. Voth and collaborators]
Coarse Graining-Parameterization-Transferability

- Need to ensure accurate simulation of other observables, which are not part of the parameterization, i.e.
- Can we improve the ”transferability” of the parametrizations?
- Recall the CKP inequality: for any observable $\phi$,

$$|\mathbb{E}_P[\phi] - \mathbb{E}_Q[\phi]| \leq ||\phi||_\infty \sqrt{2R(P | Q)}$$

$$R(P | Q) := \int \log \left( \frac{dP}{dQ} \right) dP$$

$R(P | Q)$: Loss of Information during Coarse Graining

Information-based CG parametrizations - Equilibrium

Coarse-Graining map \( \Pi : \Sigma_N \to \Sigma_M \), \( \sigma \mapsto \Pi \sigma = \eta \)

**Optimization** Parametrized CG Hamiltonian: \( \bar{H}^{\text{app}}(\eta; \theta) \)

\[
\min_\theta \mathcal{R} (\mu_N | \mu_N^{\text{app}}(\theta)) \quad \text{or} \quad \min_\theta \mathcal{R} (\mu_N^{\text{app}}(\theta) | \mu_N)
\]

Gibbs structure allows explicit calculations on \( \mathcal{R} \):

\[
\mathcal{R} \left( \bar{\mu} | \bar{\mu}^{(0)} \right) \sim \mathbb{E}_\mu [\beta (\bar{H}^{\text{app}}(\theta) - H)] + \log \frac{Z^{\text{app}}(\theta)}{Z}
\]

Optimality condition: \( \nabla_\theta \mathcal{R} = 0 \)

- **Solve using gradient methods, Newton-Raphson, etc:** M.S. Shell (2008) and (2012), Bilionis et al (2012), Zabaras et al (2013), ...
A key challenge for both equilibrium and non equilibrium CG:

- Is the parametric CG model families (for example the CG Hamiltonian $\bar{H}^{app}(\theta)$) rich enough for reliable CG predictions?
- Do we encounter “information barriers” that need costly CG terms to be overcome?
- Can we find adaptive/multi-level CG strategies to overcome them?

Next we discuss two such examples:
Model Selection–Parametrization of CG Hamiltonians

1. Cluster expansion CG Hamiltonian: \( \bar{H}_m(\eta) = \bar{H}_m^{(0)}(\eta) + \bar{H}_m^{(1)}(\eta) \)

   Multi-body terms:

   \[
   \bar{H}^{(1)}(\eta) = \beta \sum_{k_1} \sum_{k_2 > k_1} \sum_{k_3 > k_2} J_{k_1 k_2 k_3} \left( -E_1(k_1)E_2(k_2)E_1(k_3) + \ldots \right)
   \]

   Typically omitted, but essential to capture phase transitions, switching times:

