An Introduction to Malliavin calculus and its applications
Lecture II: Wiener chaos and the Ornstein-Uhlenbeck semigroup

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**Multiple stochastic integrals**

- $W = \{ W_t, t \in [0, T] \}$ is a Brownian motion defined in the canonical probability space $(\Omega, \mathcal{F}, P)$.

- $L^2_s([0, T]^n)$ is the space of symmetric square integrable functions $f : [0, T]^n \to \mathbb{R}$.

- For any $f \in L^2_s([0, T]^n)$
  \[
  \| f \|_{L^2_s([0, T]^n)}^2 = n! \int_{\Delta_n} f^2(t_1, \ldots, t_n) dt_1 \cdots dt_n,
  \]
  where \(\Delta_n = \{(t_1, \ldots, t_n) \in [0, T]^n : 0 < t_1 < \cdots < t_n < T \}\).

- If $f : [0, T]^n \to \mathbb{R}$ we define its symmetrization as
  \[
  \tilde{f}(t_1, \ldots, t_n) = \frac{1}{n!} \sum_{\sigma} f(t_{\sigma(1)}, \ldots, t_{\sigma(n)}),
  \]
  where the sum runs over all permutations $\sigma$ of $\{1, 2, \ldots, n\}$. 
The multiple stochastic integral of \( f \in L^2_s([0, T]^n) \) is defined as an iterated Itô integral:

\[
I_n(f) = n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f(t_1, \ldots, t_n) \, dW_{t_1} \cdots dW_{t_n}.
\]

We have the following property:

\[
E[I_n(f)I_m(g)] = \begin{cases} 
0 & \text{if } n \neq m \\
 n! \langle f, g \rangle_{L^2([0, T]^n)} & \text{if } n = m.
\end{cases}
\]

If \( f \in L^2([0, T]^n) \) is not necessarily symmetric we define

\[
I_n(f) = I_n(\tilde{f}).
\]
The $n$th Hermite polynomial is defined by

$$h_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2}).$$

The first Hermite polynomials are $h_0(x) = 1$, $h_1(x) = x$, $h_2(x) = x^2 - 1$, $h_3(x) = x^3 - 3x$, \ldots.

For any $g \in L^2([0, T])$ we have

$$I_n(g \otimes^n) = \|g\|^n h_n \left( \frac{\int_0^T g_t dW_t}{\|g\|} \right),$$

where $g^{\otimes^n}(t_1, \ldots, t_n) = g(t_1) \cdots g(t_n)$. 
Let \( f \in L^2_s([0, T]^n) \), and \( g \in L^2_s([0, T]^m) \). For any \( r = 0, \ldots, n \wedge m \), we define the contraction of \( f \) and \( g \) of order \( r \) to be the element of \( L^2([0, T]^{n+m-2r}) \) defined by

\[
(f \otimes_r g)(t_1, \ldots, t_{n-r}, s_1, \ldots, s_{m-r}) = \int_{[0, T]^r} f(t_1, \ldots, t_{n-r}, x_1, \ldots, x_r) g(s_1, \ldots, s_{m-r}, x_1, \ldots, x_r) dx_1 \cdots dx_r.
\]

- We denote by \( \tilde{f} \otimes_r g \) the symmetrization of \( f \otimes_r g \).
- Product of two multiple stochastic integrals

\[
l_n(f)l_m(g) = \sum_{r=0}^{n \wedge m} r! \binom{n}{r} \binom{m}{r} l_{n+m-2r}(f \otimes_r g).
\]
**Wiener Chaos expansion**

**Theorem**

\( F \in L^2(\Omega) \) can be uniquely expanded into a sum of multiple stochastic integrals:

\[
F = E[F] + \sum_{n=1}^{\infty} I_n(f_n).
\]

For any \( n \geq 1 \) we denote by \( \mathcal{H}_n \) the closed subspace of \( L^2(\Omega) \) formed by all multiple stochastic integrals of order \( n \). For \( n = 0 \), \( \mathcal{H}_0 \) is the space of constants. Then, we have the orthogonal decomposition

\[
L^2(\Omega) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n.
\]

The theorem follows from the fact that if a random variable \( G \in L^2(\Omega) \) is orthogonal to \( \bigoplus_{n=0}^{\infty} \mathcal{H}_n \), then it is orthogonal to all random variables of the form \( \left( \int_0^T g_t dW_t \right)^k \), where \( g \in L^2([0, T]) \), \( k \geq 0 \). This implies that \( G \) is orthogonal to all the exponentials \( \exp \left( \int_0^T g_t dW_t \right) \), which form a total set in \( L^2(\Omega) \). So \( G = 0 \).
Derivative operator on the Wiener chaos

Let $f \in L^2_s([0, T]^n)$. Then $I_n(f) \in D^{1,2}$, and

$$D_t I_n(f) = nI_{n-1}(f(\cdot, t))$$

**Proof**: Assume $f = g^{\otimes n}$, with $\|g\| = 1$. Let $\theta = \int_0^T g_t dW_t$. Then

$$D_t I_n(f) = D_t(h_n(\theta)) = h'_n(\theta)D_t \theta = nh_{n-1}(\theta)g_t$$

$$= ng_t I_{n-1}(g^{\otimes(n-1)}) = nI_{n-1}(f(\cdot, t)).$$

Moreover

$$E \int_0^T [D_t I_n(f)]^2 dt = n^2 \int_0^T E[I_{n-1}(f(\cdot, t))^2] dt$$

$$= n^2(n-1)! \int_0^T \|f(\cdot, t)\|_{L^2([0, T]^{n-1})}^2 dt$$

$$= nn! \|f\|_{L^2([0, T]^n)}^2$$

$$= nE[I_n(f)^2].$$
Proposition

Let $F \in L^2(\Omega)$ with the Wiener chaos expansion $F = \sum_{n=0}^{\infty} I_n(f_n)$. Then $F \in D^{1,2}$ if and only if

$$E(\|DF\|_H^2) = \sum_{n=1}^{\infty} nn!\|f_n\|^2_{L^2([0,T]^n)} < \infty,$$

and in this case

$$D_tF = \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot, t)).$$

- If $F \in D^{k,2}$, then

$$D_{t_1,\ldots,t_k}^{k} F = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)I_{n-k}(f_n(\cdot, t_1, \ldots, t_k)).$$

- As a consequence, if $F \in D^{\infty,2} := \cap_k D^{k,2}$, then (Stroock’s formula)

$$f_n = \frac{1}{n!}E(D^nF)$$
Example: \( F = W_1^3 \). Then,

\[
f_1(t_1) = E(D_{t_1} W_1^3) = 3E(W_1^2)1_{[0,1]}(t_1) = 31_{[0,1]}(t_1),
\]

\[
f_2(t_1, t_2) = \frac{1}{2} E(D_{t_1,t_2} W_1^3) = 3E(W_1)1_{[0,1]}(t_1 \vee t_2) = 0,
\]

\[
f_3(t_1, t_2, t_3) = \frac{1}{6} E(D_{t_1,t_2,t_3} W_1^3) = 1_{[0,1]}(t_1 \vee t_2 \vee t_3),
\]

and we obtain the Wiener chaos expansion

\[
W_1^3 = 3W_1 + 6 \int_0^1 \int_0^{t_1} \int_0^{t_2} dW_{t_1} dW_{t_2} dW_{t_3}.
\]
Divergence on the Wiener chaos expansion

A square integrable stochastic process \( u = \{u_t, t \in [0, T]\} \), has an orthogonal expansion of the form

\[
\begin{align*}
    u_t &= \sum_{n=0}^{\infty} I_n(f_n(\cdot, t)),
\end{align*}
\]

where \( f_0(t) = E[u_t] \) and for each \( n \geq 1 \), \( f_n \in L^2([0, T]^{n+1}) \) is a symmetric function in the first \( n \) variables.

Proposition

The process \( u \) belongs to the domain of \( \delta \) if and only if the series

\[
\begin{align*}
    \delta(u) &= \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n) \\
    \text{(1)}
\end{align*}
\]

converges in \( L^2(\Omega) \).
Proof: Suppose that $G = I_n(g)$ is a multiple stochastic integral of order $n \geq 1$, where $g$ is symmetric. Then,

$$E(\langle u, DG \rangle_H) = \int_0^T E(I_{n-1}(f_{n-1}(\cdot, t))nI_{n-1}(g(\cdot, t)))dt$$

$$= n(n-1)! \int_0^T \langle f_{n-1}(\cdot, t), g(\cdot, t) \rangle_{L^2([0,T]^{n-1})} dt$$

$$= n! \langle f_{n-1}, g \rangle_{L^2([0,T]^n)} = n! \langle \tilde{f}_{n-1}, g \rangle_{L^2([0,T]^n)}$$

$$= E \left( I_n \left( \tilde{f}_{n-1} \right) I_n(g) \right) = E \left( I_n \left( \tilde{f}_{n-1} \right) G \right).$$

If $u \in \text{Dom}\delta$, we deduce that

$$E(\delta(u)G) = E \left( I_n \left( \tilde{f}_{n-1} \right) G \right)$$

for every $G \in \mathcal{H}_n$. This implies that $I_n \left( \tilde{f}_{n-1} \right)$ coincides with the projection of $\delta(u)$ on the $n$th Wiener chaos. Consequently, the series in (1) converges in $L^2(\Omega)$ and its sum is equal to $\delta(u)$. The converse can be proved by similar arguments.
The Ornstein-Uhlenbeck semigroup

Consider the one-parameter semigroup \( \{ T_t, t \geq 0 \} \) of contraction operators on \( L^2(\Omega) \) defined by

\[
T_t(F) = \sum_{n=0}^{\infty} e^{-nt} I_n(f_n),
\]

where \( F = \sum_{n=0}^{\infty} I_n(f_n) \).

Proposition (Mehler’s formula)

Let \( W' = \{ W'_t, t \in [0, T] \} \) be an independent copy of \( W \). Then, for any \( t \geq 0 \) and \( F \in L^2(\Omega) \) we have

\[
T_t(F) = E'(F(e^{-t}W + \sqrt{1 - e^{-2t}}W'))),
\]

(2)

where \( E' \) denotes the mathematical expectation with respect to \( W' \).
Proof: Both $T_t$ and the right-hand side of (2) give rise to linear contraction operators on $L^2(\Omega)$. Thus, it suffices to show (2) when $F = \exp(\lambda W(h) - \frac{1}{2} \lambda^2)$, where $W(h) = \int_0^T h_t dW_t$ and $h \in H$ is an element of norm one, and $\lambda \in \mathbb{R}$. We have,

\[
E' \left( \exp \left( e^{-t \lambda W(h) + \sqrt{1 - e^{-2t \lambda W'(h) - \frac{1}{2} \lambda^2}}} \right) \right) = \exp \left( e^{-t \lambda W(h) - \frac{1}{2} e^{-2t \lambda^2}} \right) = \sum_{n=0}^{\infty} e^{-nt} \frac{\lambda^n}{n!} h_n(W(h)) = T_t F,
\]

because

\[
F = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} h_n(W(h))
\]

and

\[
e^{az - \frac{1}{2} a^2} = \sum_{n=0}^{\infty} \frac{a^n}{n!} h_n(z).
\]
• Mehler’s formula implies that the operator $T_t$ is nonnegative.

• The operator $T_t$ is symmetric:

$$E(GT_t(F)) = E(FT_t(G)) = \sum_{n=0}^{\infty} e^{-nt} E(I_n(f_n)I_n(g_n)).$$

• $\{T_t, t \geq 0\}$ is the semigroup of transition probabilities of a Markov process with values in $C([0, T])$, whose invariant measure in the Wiener measure. This process can be expressed in terms of a Wiener sheet $W$ as follows:

$$X_{t,\tau} = \sqrt{2} \int_{-\infty}^{t} \int_{0}^{\tau} e^{-(t-s)} W(d\sigma, ds),$$

$\tau \in [0, T], t \geq 0.$
If $F \in L^p(\Omega)$, $p > 1$ and $q(t) = e^{2t}(p - 1) + 1 > p$, $t > 0$, then
\[ \| T_t F \|_{q(t)} \leq \| F \|_p, \] (3)

Consequences:

- For any $1 < p < q < \infty$ the norms $\| \cdot \|_p$ and $\| \cdot \|_q$ are equivalent on any Wiener chaos $\mathcal{H}_n$.
- For each $n \geq 1$ and $1 < p < \infty$, the projection on the $n$th Wiener chaos is bounded in $L^p(\Omega)$. 
The generator of the Ornstein-Uhlenbeck semigroup

- The infinitesimal generator of the semigroup $T_t$ in $L^2(\Omega)$ is given by

$$LF = \lim_{t \downarrow 0} \frac{T_tF - F}{t} = \sum_{n=1}^{\infty} -nI_n(f_n),$$

if $F = \sum_{n=0}^{\infty} I_n(f_n)$.

- The domain of $L$ is

$$\text{Dom } L = \{ F \in L^2(\Omega), \sum_{n=1}^{\infty} n^2 n! \|f_n\|_2^2 < \infty \} = \mathbb{D}^{2,2}.$$
The next proposition explains the relationship between the operators $D$, $\delta$, and $L$.

**Proposition**

Let $F \in L^2(\Omega)$. Then $F \in \text{Dom } L$ if and only if $F \in D_{1,2}$ and $DF \in \text{Dom } \delta$, and in this case, we have

$$\delta DF = -LF$$

**Proof** Let $F = \sum_{n=0}^{\infty} l_n(f_n)$. For any random variable $G = l_m(g_m)$ we have, using the duality relationship

$$E(G\delta DF) = E(\langle DG, DF \rangle_H) = mm! \langle g_m, f_m \rangle_{L^2([0,T]^n)}$$

$$= E \left( G \sum_{n=1}^{\infty} nl_n(f_n) \right) = -E(GLF),$$

and the result follows easily.
The operator $L$ behaves as a second-order differential operator on smooth random variables.

**Proposition**

Suppose that $F = (F^1, \ldots, F^m)$ is a random vector whose components belong to $\mathbb{D}^{2,4}$. Let $\varphi$ be a function in $C^2(\mathbb{R}^m)$ with bounded first and second partial derivatives. Then $\varphi(F) \in \text{Dom } L$, and

$$L(\varphi(F)) = \sum_{i,j=1}^{m} \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(F) \langle DF^i, DF^j \rangle_H + \sum_{i=1}^{m} \frac{\partial \varphi}{\partial x_i}(F)LF^i.$$
Proof: Suppose first that $F \in S$ is of the form
\[ F = f(W(h_1), \ldots, W(h_n)), \]
where $f \in C^\infty_p(\mathbb{R}^n)$. Then
\[ D_t F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), \ldots, W(h_n)) h_i(t). \]
Consequently, $DF \in S_H \subset \text{Dom} \, \delta$ and we obtain
\[ \delta DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), \ldots, W(h_n)) W(h_i) \]
\[ - \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(W(h_1), \ldots, W(h_n)) \langle h_i, h_j \rangle_H, \]
which yields the desired result because $L = -\delta D$. In the general case, it suffices to approximate $F$ by smooth random variables in the norm $\| \cdot \|_{2,4}$, and $\varphi$ by functions in $C^\infty_p(\mathbb{R}^m)$, and use the continuity of the operator $L$ in the norm $\| \cdot \|_{2,2}$. 
In the finite-dimensional case ($\Omega = \mathbb{R}^n$ equipped with the standard Gaussian law),

$$L = \Delta - x \cdot \nabla$$

coincides with the generator of the Ornstein-Uhlenbeck process $\{X_t, t \geq 0\}$ in $\mathbb{R}^n$, which is the solution to the stochastic differential equation

$$dX_t = \sqrt{2}dW_t - X_t dt,$$

where $\{W_t, t \geq 0\}$ is a standard $n$-dimensional Brownian motion.
Integration-by-parts formula

Let $F \in \mathbb{D}^{1,2}$ with $E(F) = 0$. Let $f$ be a differentiable function with bounded derivative. Using that

$$ F = LL^{-1}F = -\delta(DL^{-1}F) $$

yields

$$ E[f(F)F] = -E[f(F)\delta(DL^{-1}F)] $$
$$ = -E[\langle D(f(F)), DL^{-1}F \rangle_H] $$
$$ = E[f'(F)\langle DF, -DL^{-1}F \rangle_H]. $$

If $F \in \mathcal{H}_q$, with $q \geq 1$, then $DL^{-1}F = -\frac{1}{q}DF$ and

$$ E[f(F)F] = \frac{1}{q} E[f'(F)\|DF\|_{\mathcal{H}}^2]. $$
Define for almost all $x$ in the support of $F$

$$g_F(x) = E[\langle DF, -DL^{-1}F \rangle_H | F = x]$$

- For any $f \in C^1_b(\mathbb{R})$.
  $$E[f(F)F] = E[f'(F)g_F(F)]$$

- Moreover, $g_F(F) \geq 0$ almost surely. Indeed, taking $f(x) = \int_0^x \varphi(y)dy$, where $\varphi$ is smooth and non-negative we obtain
  $$E[[\langle DF, -DL^{-1}F \rangle_H | F] \varphi(F)] \geq 0,$$
  because $xf(x) \geq 0$. 

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