An Introduction to Malliavin calculus and its applications

Lecture 3: Clark-Ocone formula

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Summer School 2014
Stochastic Integral Representation

- \( W = \{ W_t, 0 \leq t \leq T \} \) Brownian motion on the canonical probability space \((\Omega, \mathcal{F}, P)\).
- For any \( t \in [0, T] \), \( \mathcal{F}_t \) is the \( \sigma \)-algebra generated by \( \{ W_s, 0 \leq s \leq t \} \).
- A stochastic process \( u = \{ u_t, t \in [0, T] \} \) is adapted if for any \( t \in [0, T] \), \( u_t \) is \( \mathcal{F}_t \)-measurable.
- \( L^2_a \) is the space of adapted process with \( E \int_0^T u_t^2 \, dt < \infty \).

**Theorem (Itô’s Representation)**

Let \( F \in L^2(\Omega) \). Then, there exists a unique process \( u \in L^2_a \) such that

\[
F = E(F) + \int_0^T u_t \, dW_t.
\]
Clark-Ocone formula

Let $F \in \mathbb{D}^{1,2}$. Then,

$$F = E(F) + \int_0^T E(D_t F | \mathcal{F}_t) dW_t,$$

that is, $u_t = E(D_t F | \mathcal{F}_t)$.

- Clark ’70, Haussmann ’78: $F = X_t$, where $X$ is a diffusion process.
- Ocone ’84: $F \in \mathbb{D}^{1,2}$. 
Proof:

For any $v \in L^2_a$ we can write, using the duality relationship

$$E \left( F \int_0^T v_t dW_t \right) = E(F \delta(v)) = E \left( \int_0^T D_t F v_t dt \right)$$

$$= \int_0^T E[D_t F | F_t] v_t dt.$$

If we assume that $F = E(F) + \int_0^T u_t dW_t$, then by the Itô isometry

$$E \left( F \int_0^T v_t dW_t \right) = \int_0^T E(u_t v_t) dt.$$

Comparing these two expressions we deduce that

$$u_t = E(D_t F | F_t)$$

almost everywhere in $\Omega \times [0, T]$. 

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Example 1

\[ F = W_t^3. \text{ Then } D_s F = 3 W_t^2 1_{[0,t]}(s) \text{ and } \]

\[ E(D_s F | \mathcal{F}_s) = 3 E[(W_t - W_s + W_s)^2 | \mathcal{F}_s] = 3[t - s + W_s^2]. \]

Therefore,

\[ W_t^3 = 3 \int_0^t [t - s + W_s^2] dW_s. \]

Compare with Itô’s formula

\[ W_t^3 = 3 \int_0^t W_s^2 dW_s + 3 \int_0^t W_s ds. \]
Example 2

- The local time of the Brownian motion \( \{L^x_t, t \in [0, T], x \in \mathbb{R}\} \) is defined as the density of the occupation measure:

\[
\int_0^t \mathbf{1}_{\{W_s \in C\}} \, ds = \int_C L^x_t \, dx,
\]

for any Borel set \( C \in \mathcal{B}(\mathbb{R}) \).

- Formally, \( L^x_t = \int_0^t \delta_x(W_s) \, ds \). Then, if \( p_\varepsilon(x) = (2\pi \varepsilon)^{-\frac{1}{2}} e^{-x^2/2\varepsilon} \),

\[
F_\varepsilon = \int_0^t p_\varepsilon(W_s - x) \, ds \xrightarrow{L^2(\Omega)} L^x_t.
\]
We have

\[ D_r F_\varepsilon = \int_0^t p_\varepsilon'(W_s - x) D_r W_s ds = \int_r^t p_\varepsilon'(W_s - x) ds, \]

and

\[ E(D_r F_\varepsilon | F_r) = \int_r^t E(p_\varepsilon'(W_s - W_r + W_r - x) | F_r) ds \]
\[ = \int_r^t p_\varepsilon'_{s-r} (W_r - x) ds. \]

As a consequence,

\[ L_t^x = E(L_t^x) + \int_0^t \left( \int_r^t p_{s-r}'(W_r - x) ds \right) dW_r. \]
\{ L^x_t, t \geq 0, x \in \mathbb{R} \} \text{ local time of } W.

**Theorem**

$$h^{-\frac{3}{2}} \left( \int_{\mathbb{R}} (L^{x+h}_t - L^x_t)^2 dx - 4th \right) \Rightarrow 8 \sqrt{\frac{\alpha_t}{3}} \eta,$$

as \( h \to 0 \), where \( \alpha_t = \int_{\mathbb{R}} (L^x_t)^2 dx \) and \( \eta \) is a \( N(0, 1) \) random variable independent of \( W \).

- Chen, Li, Marcus, Rosen ’09 : Method of moments.
Study of the Hamiltonian for the critical attractive random polymer in dimension one:

\[ H_n = \sum_{i,j=1, i \neq j}^{n} 1\{S_i = S_j\} - \frac{1}{2} \sum_{i,j=1, i \neq j}^{n} 1\{|S_i - S_j| = 1\}, \]

where \( \{S_n, n = 0, 1, 2, \ldots\} \) is a simple random walk on the integers. Notice that

\[ H_n = \sum_{x \in \mathbb{Z}} (l^n_x - l^{n+1}_n)^2 \]

where \( l^n_x = \sum_{i=1}^{n} 1\{S_n = x\} \) is the local time of the random walk \( S_n \).
Connection with self-intersection local times

\[
\int_{\mathbb{R}} (L_t^x)^2 \, dx = \int_{\mathbb{R}} \left( \int_0^t \delta_x (W_s) \, ds \right)^2 \, dx
\]

\[
= 2 \int_0^t \int_0^v \delta_0 (W_v - W_u) \, dudv.
\]

In the same way,

\[
F_{t,h} = \int_{\mathbb{R}} (L_t^{x+h} - L_t^x)^2 \, dx
\]

\[
= 2 \int_0^t \int_0^v [\delta_0 (W_v - W_u + h) \delta_0 (W_v - W_u - h) - 2 \delta_0 (W_v - W_u)] \, dudv.
\]
Integral representation of $F_{t,h} = \int_{\mathbb{R}} (L_{t}^{x+h} - L_{t}^{x})^2 dx$

We apply Clark-Ocone formula to $F_{f,h}$.
First we need to compute the derivative of $F_{t,h}$. For $u < r < v$ we have

$$E(D_r[\delta_0(W_v - W_u + h) + \delta_0(W_v - W_u - h) - 2\delta_0(W_v - W_u)]|\mathcal{F}_r)$$
$$= E(\delta'_0(W_v - W_u + h) + \delta'_0(W_v - W_u - h) - 2\delta'_0(W_v - W_u)|\mathcal{F}_r))$$
$$= p'_{v-r}(W_r - W_u + h) + p'_{v-r}(W_r - W_u - h) - 2p'_{v-r}(W_r - W_u)$$
$$= \int_0^h [p''_{v-r}(W_r - W_u + y) - p''_{v-r}(W_r - W_u - y)]dy$$
$$= 2 \int_0^h \left[ \frac{\partial p_{v-r}}{\partial v}(W_r - W_u + y) - \frac{\partial p_{v-r}}{\partial v}(W_r - W_u - y) \right]dy,$$

where $p_\epsilon = (2\pi\epsilon)^{-\frac{1}{2}} \exp(-\frac{x^2}{2\epsilon})$. 
Integrating in $u$ and $v$ yields

$$F_{t,h} = E(F_{t,h}) + 4 \int_0^t u_{t,h}(r) \, dW_r + 4 \int_0^t \psi_h(r) \, dW_r,$$

where

$$u_{t,h}(r) = \int_0^r \int_0^h (p_{t-r}(W_r - W_u + y) - p_{t-r}(W_r - W_u - y)) \, dy \, du$$

and

$$\psi_h(r) = \int_0^r (1_{[0,h]}(W_r - W_u) - 1_{[0,h]}(W_u - W_r)) \, du.$$
Limit results

- \( E(F_{t,h}) = 4th + o(h^{\frac{3}{2}}) \).
- \( E \left( \int_0^t u_{t,h}(r)^2 dr \right) \leq Ch^4 \).
- The martingale
  \[ M^h_t = h^{-\frac{3}{2}} \int_0^t \psi_h(r) dW_r \]
  satisfies
  \[ M^h_t \Rightarrow \beta \left( \frac{4}{3} \alpha_t \right), \]  
  where \( \beta \) is a Brownian motion independent of \( W \) and
  \( \alpha_t = \int_{\mathbb{R}} (L_t^x)^2 dx \).
Sketch of the proof of (1) : The convergence

\[ M^h_t \Rightarrow \beta(\frac{4}{3}\alpha_t) \]

holds if for any \( t \geq 0 \),

\[ \langle M^h \rangle_t = h^{-3} \int_0^t \psi_h(r)^2 dr \rightarrow \frac{4}{3}\alpha_t, \quad \text{in} \quad L^2(\Omega) \quad (2) \]

and

\[ \sup_{0 \leq s \leq t} |\langle M^h, W \rangle_s| = \sup_{0 \leq s \leq t} h^{-\frac{3}{2}} \left| \int_0^s \psi_h(r) dr \right| \rightarrow 0, \quad \text{in} \quad L^2(\Omega). \quad (3) \]

In fact, (2) and (3) imply that \((W, \beta^h)\) converges in law to \((W, \beta)\), where \(\beta\) is a Brownian motion independent of \(W\), and \(\beta^h\) is such that \(M^h_t = \beta^h(\langle M^h \rangle_t)\) (asymptotic version of Ray-Knight’s theorem).
Proof of (2) :

\[
\psi_h(r) = \int_{\mathbb{R}} \left( 1_{[0,h]}(W_r - x) - 1_{[0,h]}(x - W_r) \right) L_r(x) \, dx
\]

\[
= \int_0^h \left( L_r^{W_r-y} - L_r^{W_r+y} \right) \, dy.
\]

Tanaka’s formula for the Brownian motion \( \{ W_r - W_s, 0 \leq s \leq r \} \) yields

\[
L_r^{W_r-y} - L_r^{W_r+y} = -2y - 2(W_r - y)^+ + 2(W_r + y)^+
\]

\[
+ 2 \int_0^r 1_{\{|y| > |W_r - W_s|\}} \, d\hat{W}_s,
\]

where \( d\hat{W}_s \) denote the backward stochastic Itô integral. Then,

\[
\psi_h(r) = -h^2 + 2 \int_0^h \left( (W_r + y)^+ - (W_r - y)^+ \right) \, dy
\]

\[
+ 2 \int_0^r (h - W_r - W_s)^+ \, d\hat{W}_s.
\]
It suffices to show that, in $L^2$, 

$$4h^{-3} \int_0^t \left( \int_0^r (h - |W_r - W_s|)^+ d\widehat{W}_s \right)^2 dr \rightarrow \frac{4}{3} \alpha_t.$$ 

By Itô’s formula we can write 

\[
\left( \int_0^r (h - |W_r - W_s|)^+ d\widehat{W}_s \right)^2 = 2 \int_0^r \left( \int_s^r (h - |W_r - W_u|)^+ d\widehat{B}_u \right) \\
\times (h - |W_r - W_s|)^+ d\widehat{W}_s + \int_0^r \left[ (h - |W_r - W_s|)^+ \right]^2 ds
\]

$$= I_1(r, h) + I_2(r, h).$$

The term $I_1(r, h)$ does not contribute to the above limit.
We need to show that
\[
\frac{1}{h^3} \int_0^t \int_0^r \left[ (h - |W_r - W_s|)^+ \right]^2 \, ds \, dr \to L^2(\Omega) \frac{1}{3} \alpha_t.
\]

This follows from
\[
\alpha_t = \int_{\mathbb{R}} (L^x_t)^2 \, dx = 2 \int_0^t \int_{\mathbb{R}} L^x_r L^x_{dr} \, dx = 2 \int_0^t L^W_r \, dr,
\]
\[
\int_0^t \int_0^r \left[ (h - |W_r - W_s|)^+ \right]^2 \, ds \, dr = \int_0^t \int_{\mathbb{R}} \left[ (h - |W_r - x|)^+ \right]^2 L^x_r \, dx,
\]
and
\[
\int_{\mathbb{R}} \frac{[(h - |x|)^+]^2}{h^3} \, dx = \frac{2}{3}.
\]
Define
\[
\gamma_t = \lim_{\epsilon \to 0} L^2(\Omega) \int_0^t \int_0^s p'_\epsilon(W_s - W_u) duds.
\]

\[
\gamma_t = -\frac{d}{dx} \left( \int_0^t \int_0^s \delta_x(W_s - W_u) duds \right) |_{x=0}.
\]

Proposition
\[
\gamma_t = 2 \int_0^t \left( \int_0^r p_{t-r}(W_r - W_u) du - L_{r_{W_r}} \right) dW_r.
\]
\textit{Proof :} Use Clark-Ocone formula. Set \( \gamma_t^\varepsilon = \int_0^t \int_0^s p'_\varepsilon (W_s - W_u) du ds \). Then

\[ D_r \gamma_t^\varepsilon = \int_0^t \int_0^s p''_\varepsilon (W_s - W_u) 1_{[u,s]}(r) du ds = \int_r^t \int_0^r p''_\varepsilon (W_s - W_u) du ds \]

and

\[ E(D_r \gamma_t^\varepsilon | F_r) = \int_r^t \int_0^r p''_\varepsilon + s - r (W_r - W_u) du ds \]

\[ = 2 \int_r^t \int_0^r \frac{\partial p_{\varepsilon + s - r}}{\partial u} (W_r - W_u) du ds \]

\[ = 2 \int_0^r (p_{\varepsilon + t - r} (W_r - W_u) - p_\varepsilon (W_r - W_u)) du. \]

As \( \varepsilon \) tends to zero, this converges in \( L^2([0, t] \times \Omega) \) to

\[ \int_0^r p_{t - r} (W_r - W_u) du - L^W_r. \]
The process $\gamma_t$ is a Dirichlet process (it has zero quadratic variation and infinite total variation).

**Theorem (Rogers-Walsh ’94)**

$\gamma$ has a $\frac{4}{3}$-variation in $L^2$ given by

$$\langle \gamma \rangle^\frac{4}{3}, t = K \int_0^t \left( L_r W_r \right)^\frac{2}{3} dr$$

- Proof was based on Gebelein’s inequality for Gaussian random variables.
- Alternative proof by Hu-Nualart-Song ’12 using the integral representation and ideas from fractional martingales.
CLT for the third integrated moment

Theorem

\[ h^{-2} \int_{\mathbb{R}} (L_{t+h}^x - L_t^x)^3 \, dx \Rightarrow 8\sqrt{3} \left( \int_{\mathbb{R}} (L_t^x)^3 \, dx \right)^{1/2} \eta, \]

as \( h \) tends to zero, where \( \eta \) is a \( N(0,1) \) random variable independent of \( W \).

- Hu and Nualart ’10 : Proof using Clark-Ocone formula applied to \( F_{t,h} = h^{-2} \int_{\mathbb{R}} (L_t^{x+h} - L_t^x)^3 \, dx \).
Sketch of the proof:

We have $F_{t,h} = \int_0^t \Phi_r dW_r$, where $\Phi_r = \sum_{i=1}^4 \Phi_r^{(i)}$, and

\[
\begin{align*}
\Phi_r^{(1)} &= 6 \int_{\mathbb{R}} \left(L_{r+h}^z - L_r^z\right)^2 1_{[0,h]}(W_r - z)dz \\
\Phi_r^{(2)} &= -6 \int_{\mathbb{R}} \int_0^h \left(L_{r+h}^z - L_r^z\right)^2 p_{t-r}(W_r - z - y)dydz \\
\Phi_r^{(3)} &= \frac{12h}{\sqrt{2\pi}} \int_0^r \int_{-h}^h \int_{\frac{h^2}{t-r}}^\infty p_{t-r-h^2/2}(W_r - W_s + y) \\
& \quad \times z^{-\frac{3}{2}}(1 - e^{-\frac{z}{2}})dzdyds \\
\Phi_r^{(4)} &= -\frac{12h}{\sqrt{2\pi}} \int_0^r 1_{[-h,h]}(W_r - W_s)ds \int_{\frac{h^2}{t-r}}^\infty z^{-\frac{3}{2}}(1 - e^{-\frac{z}{2}})dz
\end{align*}
\]
Limit results:

- \( \int_0^t [\Phi_r^{(3)} + \Phi_r^{(4)}] \, dW_r \) converges to \( 12\gamma t \)

- Applying Tanaka's formula to the time reversed Brownian motion \( \{W_r - W_s, 0 \leq s \leq r\} \), we decompose (up to negligible terms)
  \( \int_0^t [\Phi_r^{(1)} + \Phi_r^{(2)}] \, dW_r \) into two terms:
  - One term converges to \(-12\gamma t\)
  - A martingale term of the form

\[
48h^{-2} \int_0^t \left( \int_0^h \left( \int_0^r \left( \int_{-h}^0 (W_r - W_s - x) \, d\widehat{W}_s \right) \right) \right) \, dx \, dW_r,
\]

\[
\times \mathbf{1}_{[-h,0]}(W_r - W_\sigma - x) \, d\widehat{W}_\sigma \right) \, dx \right) \, dW_r,
\]

- to which we can apply martingale techniques.
Applications in finance

Black-Scholes model, under the risk-neutral probability

\[ dS_t = rS_tdt + \sigma S_t dW_t, \quad 0 \leq t \leq T. \]

- Any payoff \( H \geq 0 \), \( \mathcal{F}_T \)-measurable, \( E(H^2) < \infty \), can be replicated (complete market). That is, there exists a self-financing portfolio which value \( X_t \) satisfies \( X_T = H \).
- The value \( X_t \) satisfies

\[ dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t)dt. \]

if \( \Delta_t \) is the amount of shares in the portfolio.
- Applying the Itô’s representation theorem to \( e^{-rT}H \) yields

\[ e^{-rT}H = E(e^{-rT}H) + \int_0^T u_t dW_t, \]

and to have \( X_T = H \) it suffices to take \( X_0 = E(e^{-rT}H) \) and \( \Delta_t = e^{rt} \frac{u_t}{\sigma S_t} \).
Proposition

\[ u_t = e^{-rT}E(D_tH|\mathcal{F}_t) \] and the hedging portfolio of a derivative security with payoff \( H \) is given by

\[ \Delta_t = \frac{e^{-r(T-t)}}{\sigma S_t} E(D_tH|\mathcal{F}_t). \]

- In the case \( H = \Phi(S_T) \), we have \( D_tH = \Phi'(S_T)D_tS_T = \Phi'(S_T)\sigma S_T \), and

\[ \Delta_t = \frac{e^{-r(T-t)}}{S_t} E(\Phi'(S_T)S_T|\mathcal{F}_t). \]

- By the Markov property, the price of the security is of the form \( v(t, S_t) \), where

\[ v(s, x) = e^{-r(T-t)}E(\Phi(x \frac{S_T}{S_t})). \]

In that case, \( \Delta_t = v_x(t, S_t) \).
Applying the integration-by-parts formula of Malliavin calculus, we obtain

\[ \Delta_t = \frac{e^{-r(T-t)}}{S_t(T-t)} E(\Phi(S_T)(W_T - W_t)|\mathcal{F}_t). \]

This expression is well suited for Monte Carlo simulations (see Fournié, Lasry, Lebuchoux, Lions and Touzi ’99, Kohatsu-Higa and Montero ’03).
Generalization of Clark-Ocone formula:

Suppose that

$$\widetilde{W}_t = W_t + \int_0^t \theta_s \, ds,$$

where $\theta$ is adapted and $\int_0^T \theta_t^2 \, dt < \infty$ a.s.

- Suppose $E(Z_T) = 1$, where

$$Z_T = \exp \left( - \int_0^T \theta_s \, dW_s - \frac{1}{2} \int_0^T \theta_s^2 \, ds \right).$$

- Let $\frac{dQ}{dP} = Z_T$. Under $Q$, $\widetilde{W}$ is a Brownian motion (Girsanov theorem).

- In general $\mathcal{F}_{\tilde{W}}^T \subset \mathcal{F}_W^T$, with a strict inclusion.
Can we represent a random variable $\mathcal{F}_{T}^W$-measurable as a stochastic integral with respect to $\tilde{W}$ using Clark-Ocone formula?

**Theorem**

Suppose $F \in \mathbb{D}^{1,2}$, $\theta \in \mathbb{D}^{1,2}(L^2([0, T]))$, and

1. $E(Z_T^2 F^2) + E(Z_T^2 \|DF\|_H^2) < \infty$
2. $E \left( Z_T^2 F^2 \int_0^T \left( \theta_t + \int_t^T D_t \theta_s dW_s + \int_t^T \theta_s D_t \theta_s ds \right)^2 dt \right) < \infty$.

Then

$$F = E_Q(F) + \int_0^T E_Q \left( D_t F + F \int_t^T D_t \theta_s d\tilde{W}_s | \mathcal{F}_t \right) d\tilde{W}_t.$$

- Karatzas, Ocone '91 : Multidimensional case. Applications to hedging in a generalized Black-Sholes model.