An Introduction to Malliavin calculus and its applications
Lecture 4: Density formulas

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First formula for probability densities

- $\mathcal{W} = \{W_t, t \in [0, T]\}$ is a Brownian motion on the canonical probability space $(\Omega, \mathcal{F}, P)$.
- $D$ and $\delta$ are the fundamental operators in the Malliavin calculus.

**Proposition**

Let $F$ be a random variable in the space $\mathbb{D}^{1,2}$. Suppose that $\frac{DF}{\|DF\|_H^2}$ belongs to the domain of the operator $\delta$ in $L^2(\Omega)$. Then the law of $F$ has a continuous and bounded density given by

$$p(x) = E \left[ 1_{\{F > x\}} \delta \left( \frac{DF}{\|DF\|_H^2} \right) \right]. \quad (1)$$
Proof: Let $\psi$ be a nonnegative smooth function with compact support, and set $\varphi(y) = \int_{-\infty}^{y} \psi(z)dz$. Then $\varphi(F)$ belongs to $D^{1,2}$, and we can write

$$\langle D(\varphi(F)), DF \rangle_H = \psi(F)\|DF\|_H^2.$$  

Using the duality formula we obtain

$$E[\psi(F)] = E \left[ \langle D(\varphi(F)), \frac{DF}{\|DF\|_H^2} \rangle_H \right] = E \left[ \varphi(F)\delta\left(\frac{DF}{\|DF\|_H^2}\right) \right]. \quad (2)$$

By an approximation argument, Equation (2) holds for $\psi(y) = 1_{[a,b]}(y)$, where $a < b$. As a consequence, we apply Fubini’s theorem to get

$$P(a \leq F \leq b) = E \left[ \left( \int_{-\infty}^{F} \psi(x)dx \right)\delta\left(\frac{DF}{\|DF\|_H^2}\right) \right]$$

$$= \int_{a}^{b} E \left[ 1_{\{F>x\}}\delta\left(\frac{DF}{\|DF\|_H^2}\right) \right] dx$$

which implies the desired result.
Equation (1) still holds under the hypotheses $F \in \mathbb{D}^{1,p}$ and $\frac{DF}{\|DF\|_H^2} \in \mathbb{D}^{1,p'}(H)$ for some $p, p' > 1$.

Sufficient conditions are $F \in \mathbb{D}^{2,\alpha}$ and $E(\|DF\|^{-2\beta}) < \infty$ with $\frac{1}{\alpha} + \frac{1}{\beta} < 1$.

Example: Let $F = W(h)$. Then, $DF = h$ and

$$\delta \left( \frac{DF}{\|DF\|_H^2} \right) = W(h)\|h\|_H^{-2}.$$ 

As a consequence, formula (1) yields

$$p(x) = \|h\|_H^{-2} E \left[ 1_{\{F > x\}} F \right],$$

which is true because $p(x)$ is the density of $N(0, \|h\|_H^2)$. 

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Density estimates

Fix $p$ and $q$ such that $p^{-1} + q^{-1} = 1$. By Hölder’s inequality, and using (1) with $P(F < x)$, we obtain

$$p(x) \leq (P(|F| > |x|))^{1/q} \left\| \delta \left( \frac{DF}{\|DF\|_H^2} \right) \right\|_p,$$

for all $x \in \mathbb{R}$. Then, using Meyer’s inequalities we can deduce the following result.

**Proposition**

Let $q, \alpha, \beta$ be three positive real numbers such that $q^{-1} + \alpha^{-1} + \beta^{-1} = 1$. Let $F$ be a random variable in the space $D^{2,\alpha}$, such that $\mathbb{E}(\|DF\|_H^{-2\beta}) < \infty$. Then the density $p(x)$ of $F$ can be estimated as follows

$$p(x) \leq c_{q,\alpha,\beta} (P(|F| > |x|))^{1/q} \times \left( \mathbb{E}(\|DF\|_H^{-1}) + \|D^2 F\|_{L^\alpha(\Omega;H\otimes H)} \left\| \|DF\|_H^{-2} \right\|_\beta \right).$$

(4)
Example I

Set $M_t = \int_0^t u(s) dW_s$, where $u = \{u(t), t \in [0, T]\}$ is an adapted process such that $|u(t)| \geq \rho > 0$ for some constant $\rho$, $E \left( \int_0^T u(t)^2 dt \right) < \infty$, $u(t) \in D^{2,2}$ for each $t \in [0, T]$, and

$$
\lambda := \sup_{s,t \in [0,T]} \mathbb{E}(|D_s u_t|^p) + \sup_{r,s \in [0,T]} \mathbb{E} \left[ \left( \int_0^T |D_{r,s}^2 u_t|^p dt \right)^{p/2} \right] < \infty,
$$

for some $p > 3$. Then, applying Proposition 2 one can show that the density of $M_t$, denoted by $p_t(x)$ satisfies

$$
p_t(x) \leq \frac{c}{\sqrt{t}} P(|M_t| > |x|)^{\frac{1}{q}},
$$

for all $t > 0$, where $q > \frac{p}{p-3}$ and the constant $c$ depends on $\lambda$, $\rho$ and $p$. 
Example II

Consider the process \( \xi = \{\xi_{r,t}, 0 \leq r \leq t \leq T\} \) solution of the stochastic differential equation

\[
\xi_{r,t} = x + W_t - W_r + \int_r^t \int_{\mathbb{R}} h(y - \xi_{r,u}) Z(du, dy),
\]

where

- \( x \in \mathbb{R} \).
- \( W \) is a standard Brownian motion on \( \mathbb{R} \).
- \( Z \) is a Brownian sheet independent of \( W \).
- \( h \in H^2_2(\mathbb{R}) \).

(i) \( \xi \) represents the position of a particle in a random environment.

(ii) A measure valued process obtained as the limit of critical branching particles moving with the dynamics of \( \xi \) was studied by Dawson, Li and Wang '01.
For $r \leq t$ we define the conditional transition density given $Z$ by

$$p^W(r, x; t, y) = P^W(\xi_t \in dy | \xi_r = x)$$

Using the formula

$$p^W(r, x; t, y) = E^W [1_{\{\xi_r, t > y\}} \delta(u_{r,t})] ,$$

one can derive the following result:

**Lemma**

Let $c = 1 \lor \|h\|_2^2$. For any $0 \leq r < t \leq T$, $y \in \mathbb{R}$ and $p \geq 1$,

$$\|p^W(r, x; t, y)\|_{2p} \leq 2K_p \exp \left( -\frac{(x - y)^2}{64pc(t - r)} \right) (t - r)^{-\frac{1}{2}},$$

where $K_p$ is a constant depending on $p$.

This lemma has been used by Fei, Hu and Nualart ’11 to establish the Hölder continuity in space and time of the solution of the SPDE satisfied by the density of the associated limiting branching particles.
Let $F \in \mathbb{D}^{1,2}$ be such that $E[F] = 0$.

Define $g_F(x) = E[\langle DF, -DL^{-1}F \rangle_H | F = x] \geq 0$.

**Theorem (Nourdin Viens 09’)**

The law of $F$ has a density $p$ if and only if $g_F(F) > 0$ a.s. In this case the support of $p$ is a closed interval containing zero and for all $x$ in the support of $p$

$$p(x) = \frac{E[|F|]}{2g_F(x)} \exp \left( - \int_0^x \frac{ydy}{g_F(y)} \right).$$
Proof: Suppose that $F$ has a density $p$ and its support is $\mathbb{R}$. Let $\phi$ be a smooth function with compact support and $\Phi' = \phi$. Then

$$
E[\phi(F)g_F(F)] = E[\Phi(F)F] = \int_{\mathbb{R}} \Phi(x)xp(x)dx
$$

$$
= \int_{\mathbb{R}} \phi(x)\varphi(x)dx,
$$

where $\varphi(x) = \int_{x}^{\infty} yp(y)dy$. This implies

$$
\varphi(x) = p(x)g_F(x).
$$

Taking into account that $\varphi'(x) = -xp(x)$ we obtain

$$
\frac{\varphi'(x)}{\varphi(x)} = -\frac{x}{g_F(x)}.
$$

Using that $\varphi(0) = \frac{1}{2}E[|F|]$, we can write

$$
\varphi(x) = \frac{1}{2}E[|F|]\exp\left(-\int_{0}^{x} \frac{ydy}{g_F(y)}\right).
$$

which completes the proof.
Corollary

If there exist $\sigma^2_{\text{min}}, \sigma^2_{\text{max}} > 0$ such that

$$\sigma^2_{\text{min}} \leq g_F(F) \leq \sigma^2_{\text{max}}$$

a.s., then $F$ has a density $p$ satisfying

$$\frac{E[F]}{2\sigma^2_{\text{max}}} \exp \left( -\frac{x^2}{2\sigma^2_{\text{min}}} \right) \leq p(x) \leq \frac{E[F]}{2\sigma^2_{\text{min}}} \exp \left( -\frac{x^2}{2\sigma^2_{\text{max}}} \right).$$

Using Mehler’s formula and $-DL^{-1}F = \int_0^\infty e^{-t} T_t(DF) dt$, we obtain

$$g_F(F) = \int_0^\infty e^{-t} E \left[ \langle DF, E'[DF(e^{-t} W + \sqrt{1 - e^{-2t} W'})]\rangle_H \right] F \right] dt.$$ 

The above corollary can be applied if we have uniform upper and lower bounds for $\langle DF, E'[DF(e^{-t} W + \sqrt{1 - e^{-2t} W'})]\rangle_H$. 
Example

\[ F = \max_{t \in [a, b]} B^H_t - E \left( \max_{t \in [a, b]} B^H_t \right), \]

where \(0 < a < b \leq T\) and \(B^H = \{B^H_t, t \in [0, T]\}\) is a fractional Brownian motion with Hurst parameter \(H \in [1/2, 1)\). That is \(B^H\) is a zero mean Gaussian process with covariance

\[ E[B^H_t B^H_s] = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right). \]

Then, the previous corollary holds with \(\sigma_{\text{min}} = a^H\) and \(\sigma_{\text{max}} = b^H\).

- The Malliavin calculus can be developed replacing \(H\) by the closed span of the indicator functions under the inner product
  \[ \langle 1_{[0,t]}, 1_{[0,t]} \rangle_H = E[B^H_t B^H_s]. \]
  
- \(F \in \mathbb{D}^{1,2}\) and \(D_r F = 1_{[0,\tau]}(r)\), where \(\tau\) is the random point in \([0, T]\) where \(B^H\) attains its maximum.

- The lower and upper bounds for \(g_F\) follow from \(a^{2H} \leq E[B^H_t B^H_s] \leq b^{2H}\). 

Absolute continuity of random vectors

- Let $F = (F^1, \ldots, F^m)$ be such that $F_i \in \mathbb{D}^{1,2}$ for $i = 1, \ldots, m$.
- We define the *Malliavin matrix* of $F$ as the random symmetric nonnegative definite matrix
  \[
  \gamma_F = (\langle DF^i, DF^j \rangle_H)_{1 \leq i, j \leq m}.
  \]

**Theorem (Bouleau Hirsch ’91)**

*If $\det \gamma_F > 0$ a.s., then the law of $F$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^m$.***

- The proof of this theorem is based on the coarea formula and uses the techniques of geometric measure theory.
- The measure $(\det \gamma_F \circ P) \circ F^{-1}$ is always absolutely continuous, that is,
  \[
  P(F \in B, \det \gamma_F > 0) = 0
  \]
  for any $B \in \mathcal{B}(\mathbb{R}^m)$ of zero Lebesgue measure.

- In the one-dimensional case, $\gamma_F = \|DF\|^2$.  

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Smoothness of the density of random vectors

**Definition**

Let $F = (F^1, \ldots, F^m)$ be a random vector such that $F_i \in \mathbb{D}^{1,2}$ for $i = 1, \ldots, m$. We say that $F$ is nondegenerate if $E[(\det \gamma_F)^{-p}] < \infty$ for all $p \geq 2$.

Set $\partial_i = \partial / \partial x_i$, and for any multiindex $\alpha \in \{1, \ldots, m\}^k$, $k \geq 1$, we denote by $\partial_\alpha$ the partial derivative $\partial^k / (\partial x_{\alpha_1} \cdots \partial x_{\alpha_k})$.

**Lemma**

Let $\gamma$ be an $m \times m$ random matrix such that $\gamma^{ij} \in \mathbb{D}^{1,\infty}$ for all $i, j$ and $E[(\det \gamma)^{-p}] < \infty$ for all $p \geq 2$. Then $(\gamma^{-1})^{ij}$ belongs to $\mathbb{D}^{1,\infty}$ for all $i, j$, and

$$D (\gamma^{-1})^{ij} = - \sum_{k,l=1}^{m} (\gamma^{-1})^{ik} (\gamma^{-1})^{lj} D \gamma^{kl}.$$ 

**Proof:** Approximate $\gamma^{-1}$ by $\gamma_\epsilon^{-1} = (\det \gamma + \epsilon)^{-1} A(\gamma)$, where $A(\gamma)$ is the adjoint matrix of $\gamma$. 

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Proposition (Integration by parts)

Let \( F = (F^1, \ldots, F^m) \) be a nondegenerate random vector. Fix \( k \geq 1 \) and suppose that \( F_i \in D^{k+1, \infty} \) for \( i = 1, \ldots, m \). Let \( G \in D^{\infty} \) and let \( \varphi \in C^\infty_p(\mathbb{R}^m) \).

Then for any multiindex \( \alpha \in \{1, \ldots, m\}^k \), there exists an element \( H_\alpha(F, G) \in D^{\infty} \) such that

\[
E \left[ \partial_\alpha \varphi(F) G \right] = E \left[ \varphi(F) H_\alpha(F, G) \right],
\]

where the elements \( H_\alpha(F, G) \) are recursively given by

\[
H_{(i)}(F, G) = \sum_{j=1}^{m} \delta \left( G \left( \gamma_F^{-1} \right)^{ij} DF^j \right), \tag{5}
\]

\[
H_\alpha(F, G) = H_{\alpha_k}(F, H_{(\alpha_1, \ldots, \alpha_{k-1})}(F, G)). \tag{6}
\]

For any \( p > 1 \) there exist constants \( \beta, \gamma > 1 \) and integers \( n, m \) such that

\[
\|H_\alpha(F, G)\|_p \leq c_{p,q} \left\| \det \gamma_F^{-1} \right\|_\beta^m \|DF\|_{k,\gamma}^n \|G\|_{k,q}.
\]
Proof: By the chain rule we have

\[ \langle D(\varphi(F)), DF^j \rangle_H = \sum_{i=1}^m \partial_i \varphi(F) \langle DF^i, DF^j \rangle_H = \sum_{i=1}^m \partial_i \varphi(F) \gamma^i_{ij}, \]

and, consequently,

\[ \partial_i \varphi(F) = \sum_{j=1}^m \langle D(\varphi(F)), DF^j \rangle_H (\gamma^{-1}_F)^{ij}. \]

Taking expectations and using the duality relationship yields

\[ E[\partial_i \varphi(F) G] = E[\varphi(F) H_i(F, G)], \]

where \( H_i = \sum_{j=1}^m \delta \left( G \left( \gamma^{-1}_F \right)^{ij} DF^j \right). \) Notice that the continuity of the operator \( \delta \) and the previous lemma imply that \( H_i \) belongs to \( L^p \) for any \( p \geq 2. \) We can finish the proof by a recurrence argument.
Proposition

Let $F = (F^1, \ldots, F^m)$ be a nondegenerate random vector such that $F_i \in D_{m+1, \infty}$ for $i = 1, \ldots, m$. Then, $F$ has a continuous and bounded density given by

$$p(x) = E \left[ 1_{\{F > x\}} H_{\alpha}(F, 1) \right].$$

where $\alpha = (1, 2, \ldots, m)$.

- Recall that

$$H_{\alpha}(F, 1) = \sum_{j_1, \ldots, j_m=1}^m \delta \left( (\gamma^{-1}_F)^{j_1} DF^{j_1} \delta \left( (\gamma^{-1}_F)^{2j_2} DF^{j_2} \ldots \delta \left( (\gamma^{-1}_F)^{mj_m} DF^{j_m} \right) \ldots \right) \right).$$
Proof: Equality (6) applied to the multiindex $\alpha = (1, 2, \ldots, m)$ yields

$$E[\partial_\alpha \varphi(F)] = E[\varphi(F)H_\alpha(F, 1)].$$

Notice that

$$\varphi(F) = \int_{-\infty}^{F^1} \cdots \int_{-\infty}^{F^m} \partial_\alpha \varphi(x) dx.$$

Hence, by Fubini’s theorem we can write

$$E[\partial_\alpha \varphi(F)] = \int_{\mathbb{R}^m} \partial_\alpha \varphi(x) E[1_{\{F > x\}}H_\alpha(F, 1)] \, dx. \quad (7)$$

We can take as $\partial_\alpha \varphi$ any function in $C_0^\infty(\mathbb{R}^m)$ and the result follows.
Theorem (Criterion for smoothness of densities)

Let $F = (F^1, \ldots, F^m)$ be a nondegenerate random vector such that $F^i \in D_\infty$ for all $i = 1, \ldots, m$. Then the law of $F$ possesses an infinitely differentiable density.

Proof: For any multiindex $\beta$ we have (with $\alpha = (1, 2, \ldots, m)$)

$$E \left[ \partial_\beta \partial_\alpha \varphi(F) \right] = E[\varphi(F)H_\beta(F, H_\alpha(F, 1))]$$

$$= \int_{\mathbb{R}^m} \partial_\alpha \varphi(x) E \left[ 1_{\{F > x\}} H_\beta(F, H_\alpha(F, 1)) \right] \, dx.$$ 

Hence, for any $\xi \in C_0^\infty(\mathbb{R}^m)$

$$\int_{\mathbb{R}^m} \partial_\beta \xi(x)p(x)dx = \int_{\mathbb{R}^m} \xi(x)E \left[ 1_{\{F > x\}} H_\beta(F, H_\alpha(F, 1)) \right] \, dx.$$ 

Therefore $p(x)$ is infinitely differentiable, and for any multiindex $\beta$ we have

$$\partial_\beta p(x) = (-1)^{|\beta|} E \left[ 1_{\{F > x\}} H_\beta(F, (H_\alpha(F, 1)) \right].$$
Density formulas using the Riesz transform

1. Let $Q_m$ be the fundamental solution to the Laplace equation $\Delta Q_m = \delta_0$ on $\mathbb{R}^m$, $m \geq 2$. That is,

$$Q_2(x) = a_2^{-1} \ln \frac{1}{|x|}, \quad Q_m(x) = a_m^{-1}|x|^{2-m}, \quad m > 2,$$

where $a_m$ is the area of the unit sphere in $\mathbb{R}^m$.

2. We know that

$$\partial_i Q_m(x) = -c_m \frac{x_i}{|x|^m},$$

where $c_m = 2(m-2)/a_m$ if $m > 2$ and $c_2 = 2/a_2$.

**Lemma**

Any function $\varphi$ in $C^1_0(\mathbb{R}^m)$ can be written as

$$\varphi(x) = \nabla \varphi \ast \nabla Q_m(x).$$

**Proof:**

$$\nabla \varphi \ast \nabla Q_m(x) = \varphi \ast \Delta Q_m(x) = \varphi(x).$$
Theorem (Bally-Caramellino ’11, ’12)

Let $F$ be an $m$-dimensional non-degenerate random vector whose components are in $\mathbb{D}^{2,\infty}$. Then the law of $F$ admits a continuous and bounded density $p$ given by

$$p(x) = \sum_{i=1}^{m} E \left( \partial_i Q_m(F - x) H_{(i)}(F, 1) \right).$$

Recall that

$$H_{(i)}(F, 1) = \sum_{j=1}^{m} \delta \left( (\gamma_F^{-1})_{ij} D F^j \right).$$
Proof : Let \( \varphi \in C_0^1(\mathbb{R}^m) \). Applying the previous lemma, Fubini’s theorem, and the integration by parts formula we get

\[
E (\varphi(F)) = \sum_{i=1}^{m} \int_{\mathbb{R}^m} \partial_i Q_m(y) E (\partial_i \varphi(F - y)) \, dy \\
= \sum_{i=1}^{m} \int_{\mathbb{R}^m} \partial_i Q_m(y) E (\varphi(F - y) H(i)(F, 1)) \, dy \\
= \sum_{i=1}^{m} \int_{\mathbb{R}^m} \varphi(y) E (\partial_i Q_m(F - y) H(i)(F, 1)) \, dy.
\]

The use of Fubini’s theorem needs to be justified by showing that all the functions are integrable. Assume that the support of \( \varphi \) is included in the ball \( B_R(0) \) for some \( R > 1 \). Then,

\[
E \int_{\mathbb{R}^m} |\partial_i Q_m(y) \partial_i \varphi(F - y)| \, dy \leq |\partial_i \varphi|_\infty E \int_{\{y: |F| - R \leq |y| \leq |F| + R\}} |\partial_i Q_m(y)| \, dy \\
\leq C_m |\partial_i \varphi|_\infty E \int_{|F| - R}^{|F| + R} \frac{r}{r_m} r^{m-1} \, dr \\
= 2C_m R |\partial_i \varphi|_\infty < \infty.
\]
Lemma (Stroock)

For any \( p > m \) there exists a constant \( c \) depending only on \( m \) and \( p \) such that

\[
\|p\|_{\infty} \leq c \left( \max_{1 \leq i \leq m} \|H(i)(F, 1)\|_p \right)^m.
\]

Proof: Suppose \( m \geq 3 \). From

\[
p(x) = \sum_{i=1}^{m} E \left( \partial_i Q_m(F - x) H(i)(F, 1) \right)
\]

Hölder’s inequality with \( \frac{1}{p} + \frac{1}{q} = 1 \) and the estimate

\[
|\partial_i Q_m(F - x)| \leq c_m |F - x|^{1-m}
\]

yields

\[
p(x) \leq mc_m A \left( E[|F - x|^{(1-m)q}] \right)^{1/q},
\]

where \( A = \max_{1 \leq i \leq m} \|H(i)(F, 1)\|_p \).
i) Suppose first that \( p \) is bounded by a constant \( M \). We can write for any \( \epsilon > 0 \),

\[
E[|F - x|^{(1-m)q}] \leq \epsilon^{(1-N)q} + \int_{|z-x|\leq \epsilon} |z - x|^{(1-N)q} p(x) \, dx
\]

\[
\leq \epsilon^{(1-N)q} + C_N \epsilon^{\frac{p-N}{p-1}} M.
\]

Therefore,

\[
M \leq ANk_N \left( \epsilon^{1-N} + C_N^\frac{1}{q} \epsilon^{\frac{p-N}{p}} M^{\frac{1}{q}} \right).
\]

Now we minimize with respect to \( \epsilon \) and we obtain

\[
M \leq AC_{N,p} M^{1-\frac{1}{N}}
\]

for some constant \( c_{N,p} \), which implies \( M \leq c_{N,p} A^N \).

ii) If \( p \) is not bounded, we apply the procedure to \( p \ast \psi_\delta \), where \( \psi_\delta \) is an approximation of the identity and let \( \delta \) tend to zero at the end.