Classical limit for a system of random non-linear Schrödinger equations

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We are interested in random non-linear semiclassical Schrödinger equations of the form

\[ i\varepsilon \partial_t \psi^\varepsilon = -\frac{\varepsilon^2}{2} \Delta \psi^\varepsilon + V^\varepsilon \psi^\varepsilon + U^\varepsilon [\psi^\varepsilon] \psi^\varepsilon, \quad t > 0, \quad x \in \mathbb{R}^d, \]

where \( \varepsilon \ll 1 \) and

- \( V^\varepsilon \) is a random potential
- \( U^\varepsilon \) is non-linear.

The motivation comes from

- high-frequency wave propagation in random media
- or the quantum dynamics of self-interacting particles subject to impurities.
Our main questions are

▶ Can the limit $\varepsilon \to 0$ be rigorously investigated?
▶ Are self-averaging effects observed in the linear case still present when adding a (nice) non-linearity?

The last item is crucial for the resolution of some inverse problems in random media: in the linear case, quadratic functions of the random wavefunction $\psi^\varepsilon$ converge to deterministic quantities that can be used for the inversion.

We will focus here on the explicit case of the quantum dynamics.
We consider the following system

\[ i\varepsilon \partial_t \psi_i^\varepsilon = -\frac{\varepsilon^2}{2} \Delta \psi_i^\varepsilon + V^\varepsilon \psi_i^\varepsilon + U^\varepsilon \psi_i^\varepsilon, \quad t > 0, \quad x \in \mathbb{R}^3, \]

\[ \psi_i^\varepsilon (t = 0) = \psi_{0,i}^\varepsilon, \quad i \in \mathbb{N} \]

where

- \( \varepsilon \equiv \) Planck constant
- \( \psi_i^\varepsilon \) wave function associated with the state \( i \)
- \( V^\varepsilon \) random potential
- \( U^\varepsilon \) non-linear interaction potential in the mean field approximation (this essentially means many weak interactions)
The non-linearity is given by the Hartree potential
\[ U^\varepsilon = \frac{1}{4\pi |x|} * n^\varepsilon, \quad n^\varepsilon(t, x) = \sum_{i \in \mathbb{N}} \rho_i^\varepsilon |\psi_i^\varepsilon(t, x)|^2 \]
with \( \rho_i^\varepsilon > 0 \) and \( \sum_{i \in \mathbb{N}} \rho_i^\varepsilon = 1 \).

The random potential is supposed to have a weak amplitude and to oscillate in time and space at a frequency \( 1/\varepsilon \):
\[ V^\varepsilon(t, x) = \sqrt{\varepsilon} V \left( \frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right) \]

What is the dynamics as \( \varepsilon \rightarrow 0 \)?
Outline

- The linear random case $U^\varepsilon \equiv 0$
- The deterministic non-linear case $V^\varepsilon \equiv 0$
- Main result for the random non-linear case $U^\varepsilon \neq 0$ and $V^\varepsilon \neq 0$
- Convergence to the Vlasov-Poisson-Boltzmann equation
- Some ingredients of the proof
The linear case $U^\varepsilon \equiv 0$

Now a classical problem (Bal, Erdös, Komorowski, Fannjiang, Gomez, Lukkarinen, Papanicolaou, Poupaud, Ryzhik, Spohn, Vasseur, Yau...).

Define the density operator (positive, hermitian, trace class)

$$ \varrho^\varepsilon(t) = \sum_{i \in \mathbb{N}} \rho_i^\varepsilon |\psi_i^\varepsilon(t)><\psi_i^\varepsilon(t)| $$

and the corresponding Wigner transform

$$ W^\varepsilon(t, x, k) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ik \cdot y} \rho^\varepsilon \left( t, x - \frac{\varepsilon}{2} y, x + \frac{\varepsilon}{2} y \right) dy. $$

($\rho^\varepsilon$ integral kernel of $\varrho^\varepsilon$)
The linear case $U^\varepsilon \equiv 0$

Under appropriate assumptions on $V$ (e.g. stationary ergodic Markov) and $W^\varepsilon(t = 0)$ (e.g. $W^\varepsilon$ bounded in $L^2$), then

$$W^\varepsilon \to W, \quad \text{weakly in } L^2 \text{ and in probability},$$

where $W$ satisfies the radiative transfer equation

$$\partial_t W + k \cdot \nabla_x W = \mathcal{L}W, \quad t > 0, \quad (x, k) \in \mathbb{R}^3 \times \mathbb{R}^3$$

$$W(t = 0) = W_0.$$ 

- $\mathcal{L}$ linear collision operator depending on the correlation function of $V$

$$\mathcal{L}W(k) = \int_{\mathbb{R}^3} \frac{dp}{(2\pi)^3} \hat{R} \left( \frac{|k|^2 - |p|^2}{2}, p - k \right) (W(p) - W(k))$$

- $W_0$ weak limit of the initial Wigner transform
The linear case $U^\varepsilon \equiv 0$

**Remark:** The technique of proof heavily depends on the assumptions of the random potential

- When $V \equiv V(t, x)$ is Markovian, or (sub)Gaussian, proof relatively straightforward using martingale techniques
- When $V \equiv V(x)$, Feynman diagrams (Erdös-Yau), only convergence of $\mathbb{E}\{W^\varepsilon\}$
The non-linear case $V^\varepsilon \equiv 0$

Lions-Paul 93. Under appropriate assumptions on the initial condition ($W^\varepsilon$ bounded in $L^2$ and kinetic energy uniformly bounded),

$$W^\varepsilon \rightarrow W, \text{ weakly in } L^2$$

where $W$ satisfies the Vlasov-Poisson equation

$$\partial_t W + k \cdot \nabla_x W - \nabla_x U \cdot \nabla_k W = 0, \quad t > 0, \quad (x, k) \in \mathbb{R}^3 \times \mathbb{R}^3$$

$$U = \frac{1}{4\pi|x|} * n, \quad n(t, x) = \int_{\mathbb{R}^3} W(t, x, k) dk$$

$W(t = 0) = W_0.$
The non-linear case $V^\varepsilon \equiv 0$

Natural estimates yield that $n^\varepsilon$ bounded in $L^1$. This does not give enough regularity on $U^\varepsilon$ to pass to the limit in the non-linear term.

Better estimates are obtained from Lieb-Thirring inequalities:

$$\|n^\varepsilon(t)\|_{L^q} \leq C_q \left( \text{Tr}(\varrho^\varepsilon(t)^2) \right)^{\theta/2} \left( \mathcal{E}_k^\varepsilon(t) \right)^{1-\theta} \varepsilon^{2\theta-2}, \quad \theta \in [0, 1],$$

with

$$\theta = \frac{3}{2q} - \frac{1}{2} \quad \text{and} \quad \frac{7}{5} \leq q \leq 3$$

and

$$\mathcal{E}_k^\varepsilon(t) = \int_{\mathbb{R}^6_{xk}} |k|^2 W^\varepsilon(t, x, k) \, dx \, dk = \varepsilon^2 \text{Tr} \, \sqrt{-\Delta} \varrho^\varepsilon(t) \sqrt{-\Delta}$$

$$= \sum_{i \in \mathbb{N}} \rho_i^\varepsilon \varepsilon \|\nabla \psi_i^\varepsilon(t)\|_{L^2}^2.$$
The non-linear case $V^\varepsilon \equiv 0$

This gives

$$\| n^\varepsilon(t) \|_{L^{7/5}} \leq C \left( \text{Tr}(\varrho^\varepsilon(t)^2) \right)^{\frac{2}{7}} (\mathcal{E}^\varepsilon_k(t))^{\frac{3}{7}} \varepsilon^{-\frac{6}{7}}.$$ 

If $\| W^\varepsilon(t = 0) \|_{L^2}$ is uniformly bounded, then $\text{Tr}(\varrho^\varepsilon(t)^2) \leq C \varepsilon^3$ and

$$\| n^\varepsilon(t) \|_{L^{7/5}} \leq C (\mathcal{E}^\varepsilon_k(t))^{\frac{3}{7}}.$$ 

Then, since it is direct to show that $\mathcal{E}^\varepsilon_k(t) \leq C$, there is enough regularity to pass to the limit.
Results for the random non-linear case

Suppose now that $V^\varepsilon \neq 0$ and $U^\varepsilon \neq 0$.

Combining the last two results, one may expect that $W^\varepsilon$ converges to the solution to the Vlasov-Poisson-Boltzmann equation

$$\partial_t W + k \cdot \nabla_x W - \nabla_x U \cdot \nabla_k W = \mathcal{L}W.$$  

Main questions:

- Can this be proved rigorously?
- Do we still observe self-averaging despite the nonlinearity?
Results for the random non-linear case

Main assumptions:

- $n^\varepsilon(t = 0)$ is uniformly bounded in $L^1$
- The initial kinetic energy is uniformly bounded $\mathcal{E}^\varepsilon_k(0) \leq C$
- $W^\varepsilon(t = 0)$ is uniformly bounded in $L^2$
- The limit $W_0$ verifies the conditions of Lions-Perthame 94 that ensure uniqueness for the Vlasov-Poisson equation
- $V$ is a stationary ergodic Markov process
Results for the random non-linear case

Theorem

\[ W^\varepsilon \rightarrow W, \quad \text{weakly in } L^2 \text{ and in probability}, \]

where \( W \) satisfies the Vlasov-Poisson-Boltzmann equation

\[ \partial_t W + k \cdot \nabla_x W - \nabla_x U \cdot \nabla_k W = \mathcal{L}W. \]

Moreover, we have convergence for the density and the current:

\[
\int_{\mathbb{R}^3} W^\varepsilon(t, x, k) dk \rightarrow \int_{\mathbb{R}^3} W(t, x, k) dk, \quad \text{weakly in } L^{7\over 5} \text{ and in proba.}
\]

\[
\int_{\mathbb{R}^3} k W^\varepsilon(t, x, k) dk \rightarrow \int_{\mathbb{R}^3} k W(t, x, k) dk, \quad \text{weakly in } L^{7\over 6} \text{ and in proba.}
\]
Remarks

▶ Classical results in the linear case yield, for all smooth test functions $\varphi$:

$$
\int_{\mathbb{R}^6_{xk}} \varphi(x, k) W^\varepsilon(t, x, k) dk \rightarrow \int_{\mathbb{R}^6_{xk}} \varphi(x, k) W(t, x, k) dk
$$

We have here the stronger result (independent of the non-linearity, it’s all based on the energy) that the density $n^\varepsilon$ and the current are self-averaging.

▶ The self-averaging property of the linear case holds for “nice” non-linearities (e.g. $(−\Delta)^{-s} n^\varepsilon$, $s$ sufficiently large), focusing or defocusing
Ingredients of the proof

- Martingale formulation and perturbed test functions
- The problem is again to pass to the limit in the non-linear term
- In the linear case, one usually only has an estimate for $n^\varepsilon$ in $L^1$ (even for mixed states). Not enough.
- We need an estimate on the kinetic energy
  
  This is the key point.

Roughly:

$$E_k(t) = E_k(0) + \sqrt{\varepsilon} \int_0^t \int_{\mathbb{R}^3} \frac{\partial}{\partial s} \left[ V \left( \frac{s}{\varepsilon}, \frac{x}{\varepsilon} \right) \right] n^\varepsilon(s, x) ds dx$$

The term on the right blows up when $\varepsilon \to 0!$ But still, the kinetic energy in the limiting equation should be bounded if the initial condition is.
Ingredients of the proof

- Hence, $\mathcal{E}_k(t)$ might not be bounded pathwise, but remembering that $W^\varepsilon - \mathbb{E}\{W^\varepsilon\} \to 0$, one can expect actually that
  \[
  \mathbb{E}\{\mathcal{E}_k(t)\} \leq C.
  \]

- The bound is obtained by using the martingale formulation along with well-chosen perturbed test functions.

- Calculations give
  \[
  \mathbb{E}\{\langle W^\varepsilon(t), |k|^2 \rangle \} \leq C + C\sqrt{\varepsilon} + C \int_0^t \mathbb{E}\{\langle W^\varepsilon(s), |k|^2 \rangle \} ds
  \]
  and the estimate follows from the Gronwall Lemma. The difficulty above is to obtain a sublinear estimate.
Ingredients of the proof

The other ingredients are classical:

- Tightness and convergence by the martingale formulation
- Skohorod representation to handle the non-linear term
- Self-averaging and the density and the current are consequences of the energy estimate
Open questions

- More general hypotheses on $V(t, x)$?
- Long-range dependence?
- What about the case $V \equiv V(x)$?
- Stronger non-linearities?