Recall: $A_{m \times n}$

$B$ is left-inverse of $A$ provided $BA = I$

$C$ is right-inverse of $A$ provided $AC = I$

$E$ is inverse of $A$ provided $AE = I$ and $EA = I$

Note: we'll show that when $A$ is square

$E$ is left-inverse of $A$

iff

$E$ is right-inverse of $A$

iff

$E$ is inverse of $A$.

Usefulness of inverses:

If $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, find a matrix $X$

\[ AX = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \]

s.t. $AX = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$.

Given that $A$

has an inverse $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

If $AX = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

Result by $A^{-1}$ to get $A^{-1}AX = A^{-1} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

So only possible solve is $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, and $A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.
So $X = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}$ is the soln.

**Proposition**

Suppose $A$ in $\mathbb{R}^{m \times n}$ has an inverse. Then the inverse is unique.

**Proof.** Spse $B$, $C$ are inverses of $A$.

Consider $BAC$.

On one hand this is $(BAC)C = IC = C$.

On other hand this is $B(AC) = BI = B$.

Since matrix mult. is commutative.

So $B = C$. $\blacksquare$

If $A$ is square and has an inverse we say $A$ is invertible and write $A^{-1}$ for its unique inverse.

Eg. $P$ a perm matrix

$P$ is invertible $\iff P^{-1} = P^T$
Basic Properties of invertible matrices

\( A_{n \times n}, B_{m \times n} \)

\( A \text{ inv}, B \text{ inv} \implies AB \text{ inv}, \text{ and } (AB)^{-1} = B^{-1}A^{-1} \)

\( A \text{ inv} \implies A^T \text{ inv} \text{ and } (A^T)^{-1} = (A^{-1})^T \)

\( A \text{ inv} \implies A^H \text{ inv} \text{ and } (A^H)^{-1} = (A^{-1})^H \)

How can we tell if a matrix \( A \) is invertible?

This was a very important question in development of matrix theory.

Answer. The determinant.

The determinant is a special func from \( n \times n \)

matrices to \( \mathbb{R} \) (or \( \mathbb{C} \)) with the

property that \( \det A \neq 0 \text{ if and only if } A \text{ has an inverse.} \)

Let's define the determinant.

Some preliminaries.

Given a permutation \( \sigma(1), \sigma(2), \ldots, \sigma(n) \)

then inversion of \( \sigma \) is a pair \( i,j \) with \( i<j \)

but \( \sigma(i) > \sigma(j) \).
Ex.

\( 1 \ 4 \ 3 \ 2 \ 5 \) \hspace{1cm} \text{has 3 inversions}

\( 5 \ 4 \ 3 \ 2 \ 1 \) \hspace{1cm} \text{has \(4+3+2+1\) inversions.}

The sign of a permutation \( \sigma \) is \((-1)^{\text{# of inv.}}\).

So \( \text{sign}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ has an even # of inversions} \\ -1 & \text{if } \sigma \text{ has an odd # of inversions.} \end{cases} \)

\[
\text{defn} \quad A = \begin{bmatrix} a_{ij} \end{bmatrix}_{n \times n}
\]

\[
\text{det} \ A = \sum_{\sigma \in S_n} \text{sign}(\sigma) \quad a_{\sigma(1)}a_{\sigma(2)} \cdots a_{\sigma(n)}
\]

Ex:

\[
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}
\]

\[
\text{det} \ A = \text{sign}(1 \ 2) \ a_{11}a_{22} + \text{sign}(2 \ 1) \ a_{12}a_{21}
\]

\[
= a_{11}a_{22} - a_{12}a_{21}
\]

\[
B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}
\]

\[
\text{det} \ B = b_{11}b_{22}b_{33} + b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32}
\]

\[
- b_{12}b_{21}b_{33} - b_{11}b_{23}b_{32} - b_{13}b_{22}b_{31}
\]

Corr. per

\( 1 \ 2 \ 3 \)

\( 2 \ 3 \ 1 \)

\( 3 \ 1 \ 2 \)

\( 1 \ 3 \ 2 \)
$n=4$ \quad \det A \text{ has } 4! = 24 \text{ terms. Ouch!}

Some determinants are easy to calculate.

**Eg. 1.** A lower triangular:

\[
A = \begin{bmatrix}
a_{11} & a_{12} & 0 \\
* & \ddots & * \\
* & & a_{nn}
\end{bmatrix}
\]

Only nonzero term in det expansion of $A$

is \[\text{sgn}(1 \cdot \ldots \cdot n) a_{11} a_{22} \cdots a_{nn} = a_{11} a_{22} \cdots a_{nn} .\]

2. \[R = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

\[
\det R = \text{sgn}(n \ 1 \cdot \ldots \cdot n-1 \ 3 \ 2) \ 1 \ 1 \ \cdots \ 1 \\
= (-1)^{\binom{n}{2}} \\
= (-1)^{\frac{n(n-1)}{2}}
\]

3. Det of matrix with row or col of 0's

is \[0.\]