Last time saw:
\[ \det \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \text{signed volume} \]

\[ \det \begin{bmatrix} d_1 & a & b \\ 0 & d_2 & c \\ 0 & 0 & d_3 \end{bmatrix} = ? \]

Volume of parallelepiped
\[ \text{Spanned by } \begin{bmatrix} d_1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ d_2 \\ 0 \end{bmatrix}, \begin{bmatrix} b \\ c \\ d_3 \end{bmatrix} \]

\[ = d_3 \cdot \text{area of base} \]
\[ = d_3 \cdot \text{area of parallelogram spanned by } \begin{bmatrix} a \\ d_2 \\ 0 \end{bmatrix}, \begin{bmatrix} d_1 \\ 0 \\ 0 \end{bmatrix} \]
\[ = d_3 \cdot d_2 \cdot d_1 = \det \begin{bmatrix} d_1 & 0 & b \\ 0 & d_2 & c \\ 0 & 0 & d_3 \end{bmatrix} \]
9.2

In general:

Any \( v_1, v_2, \ldots, v_n \) in \( \mathbb{R}^n \)
can be rotated to \( u_1, u_2, \ldots, u_n \)
of form \[
\begin{bmatrix}
u_1 & v_2 & \cdots & v_n
\end{bmatrix}
\begin{bmatrix}
d_1 \\
d_2 \\
\vdots \\
d_n
\end{bmatrix}
\]
So \[
\begin{bmatrix}
u_1 & v_2 & \cdots & v_n
\end{bmatrix}
= \begin{bmatrix}
u_1 & \cdots & u_n
\end{bmatrix}
\]
\[
\det R \det \begin{bmatrix}
u_1 & \cdots & v_n
\end{bmatrix} = \det \begin{bmatrix}
u_1 & \cdots & u_n
\end{bmatrix} = \text{signed volume of parallelepiped}
\]
\[
\text{Gen by } u_1, \ldots, u_n
\]
\[
= \text{signed volume of parallelepiped}
\]
\[
\text{Gen by } v_1, \ldots, v_n
\]

Upshot:

\[
\det \begin{bmatrix}
u_1 & \cdots & v_n
\end{bmatrix} = \text{signed of parallelepiped determined by} \begin{bmatrix}
u_1 & \cdots & v_n
\end{bmatrix}
\]

From Calc 3:

\[
\iiint_{\text{Region N}} f(x, y, z) \, dx \, dy \, dz = \iiint_{\text{Region N'}} f(u, v, w) \, \det J \, du \, dv \, dw
\]

Substitution

\[
\begin{align*}
u &= g_1(x, y, z) \\
v &= g_2(x, y, z) \\
w &= g_3(x, y, z)
\end{align*}
\]

where \( J = \begin{bmatrix}
\frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\
\frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \\
\frac{\partial g_3}{\partial x} & \frac{\partial g_3}{\partial y} & \frac{\partial g_3}{\partial z}
\end{bmatrix} \)
Why?

\[ \text{Signed new volume} = \det \begin{vmatrix} \frac{\partial s_1}{\partial x} & \frac{\partial s_1}{\partial y} & \frac{\partial s_1}{\partial z} \\ \frac{\partial s_2}{\partial x} & \frac{\partial s_2}{\partial y} & \frac{\partial s_2}{\partial z} \\ \frac{\partial s_3}{\partial x} & \frac{\partial s_3}{\partial y} & \frac{\partial s_3}{\partial z} \end{vmatrix} \]

Determinants Count Lattice paths

Standard problem:
Bobby's home is located 3 blocks west and 5 blocks south of his school. He walks to school by always walking a block north or a block east. How many different routes to school are there?
In general a route corresponds to a string of 3 E's and 5 N's in some order.

So \# of routes = \binom{3+5}{3} = \binom{8}{3} \quad \text{among 8 blocks choose 3 to go east on.}

A lattice path in \( \mathbb{R}^2 \) is one that starts at a point \((i,j)\), ends at a point \((e,d)\) and at each stage goes one unit to right or one unit up.

\# of lattice paths from \((a,b)\) to \((e,d)\) is 

\[ \binom{e-d}{C-a} \]
Now let's consider $n$ pairs of initial pts $(a_1, b_1), \ldots, (a_n, b_n)$ and $n$ pairs of terminal points $(x_1, y_1), \ldots, (x_n, y_n)$.

A nest of lattice paths consists of $n$ lattice paths whose initial pts are $(a_1, b_1), \ldots, (a_n, b_n)$ $\&$ terminal pts are $(x_1, y_1), \ldots, (x_n, y_n)$ in some order.

A nest from $(0,0)$ to $(-1,1)$

A nest is non-intersecting if no pair of paths meet.

A non-intersecting nest.
9.6

**Question:** How many nonintersecting nests from \((a, b)_1 \ldots (a_n, b_n)

to \((x_1, y_1) \ldots (x_n, y_n)\)?

**Answer:** A determinant!

**Example:**

\[
A = \begin{bmatrix}
(a_1, b_1) & (2, 2) \\
(0, 0) & (y) \\
(-1, 1) & (y) \\
(4) & (y)
\end{bmatrix}
\]

\[
\det A = \det \begin{bmatrix}
1 & 6 \\
4 & 1
\end{bmatrix} = -20
\]

So \(\# \text{ of nonint. nests from}

\((4, 0), (-1, 1) \to

(0, 4), (2, 2)\)

is \(|-20| = 20\).

See if you could find these 20.