Enduring Understandings

- Algebra is useful, universal and relevant in everyday life and in K-12 mathematics education.
- Algebra is a living organism; algebraic tools are developed to enable better understanding of the world around us; the ability to think abstractly is a valuable asset.
- Algebraic properties, rules, and concepts make sense.
- Mathematics is a language communicated through specialized vocabulary and symbols used to represent mathematical ideas, generalizations and relationships.
- Symmetry plays a key role in science, math, engineering and culture

Permutations: different views Recall: A permutation of X is a one-to-one correspondence $f : X \to X$.

Example: A permutation of $X = \{1, 2, 3, 4, 5, 6\}$ is given by : $f(1) = 4$, $f(2) = 3$, $f(3) = 2$, $f(4) = 5$, $f(5) = 1$, $f(6) = 6$

2-line notation

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 5 & 1 & 6 \end{pmatrix}$$

Diagram of permutation The diagram of f has nodes $1, 2, \ldots, n$ and arc $i \to f(i)$.

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Properties of diagram
Since f is a permutation each node has exactly one arc entering, and one arc leaving it.

Consequently, the diagram is a disjoint union of cycles.

Cycle notation

We use the shorthand $(i_1, i_2, \ldots, i_k)$ for the colorred cycle:
$i_1 \to i_2 \to i_3 \to \cdots \to \cdots i_k \to i_1$.

And for a disjoint union of cycles:

Cycle notation This way every permutation can be written as a product of disjoint cycles.

Example:

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 4 & 5 & 3 & 1 & 2 \end{pmatrix}$$

Draw the diagram of f.

Express f as a product of disjoint cycles

Express f as a product of disjoint cycles in another way.

Express f^{-1} as a product of disjoint cycles

Cycle notation

Now let $f = (1, 2, 7, 4)(3, 9)$

What's $f(1)$?, $f(5)$?

What's the 2-line notation for f?

What's $f^2(1)$?

What the smallest positive integer k with $f^k = \mathrm{id}$?

Disjoint cycles commute
Note if two cycles have no elements in common they commute:

E.g. $(1, 2, 4, 5)(7, 8)$ and $(7, 8)(1, 2, 4, 5)$ represent the same permutation.

Cycle notation:inverses
The inverse of the cycle $(i_1, i_2, \ldots, i_k)$ is $(i_1, i_k, i_{k-1} \ldots i_2)$.

The inverse of a product of **disjoint** cycles $c_1 c_2 \ldots, c_r$ is $c_1^{-1} c_2^{-1} \cdots c_r^{-1}$.

Composition of cycles Example:
Write $(1,5)(2,3,5)(7,9)(1,2,4)(7,8)(1,4,6)$ as a product of disjoint cycles.
Always move right to left

- 1 goes to 4, which goes to 4, which goes to 1 which goes to 1 which goes to 1 which goes to 5. So in the composition $1 \rightarrow 5$.

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- Now let's see where 5 goes: 5 goes to 5, which goes to 5, which goes to 5, which goes to 5, which goes to 2, which goest to 2. So in the composition $5 \to 2$.

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- 2 ends up at 4;

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- 2 ends up at 4;
- 4 ends up 6

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- 2 ends up at 4;
- 4 ends up 6
- 6 ends up at 3

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- 2 ends up at 4;
- 4 ends up 6
- 6 ends up at 3
- 3 ends up at 1.

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- 1 goes to 4, which goes to 4, which goes to 1 which goes to 1 which goes to 1 which goes to 5. So in the composition $1 \rightarrow 5$.
- Now let's see where 5 goes: 5 goes to 5, which goes to 5, which goes to 5, which goes to 5, which goes to 2, which goest to 2. So in the composition $5 \rightarrow 2$.
- 2 ends up at 4;
- 4 ends up 6
- 6 ends up at 3
- 3 ends up at 1.
- So $(1,5,2,4,6,3)$ is a cycle of the composition.

Composition of cycles Example:
Write $(1,5)(2,3,5)(7,9)(1,2,4)(7,8)(1,4,6)$ as a product of disjoint cycles.

Repeat with other symbols. Get $(7,8,9)$ is another cycle.

So composition is $(1,5,2,4,6,3)(7,8,9)$.

Other examples

- $(1,2)(3,4)(1,3)(2,4)$
- $(1,2)(1,3)(1,4)(1,5)(1,6)$
- $(1,2,3,4,5)^2$
- $[(1,2,3)(1,2)]^3$
- Cayley table for all permutations of $\{1,2,3\}$.

Two kinds of permutations
Just like there are two types of integers (even or odd), there are two types of permutations (even or odd).

An inversion of a permutation f on $\{1, 2, \dots, n\}$, is a pair of integer i and j with $I < j$ but $f(i) > f(j)$.

Example: $(1, 3)$ is an inversion of the permutation corresponding to $2, 4, 1, 3$ because the numbers in positions 1 and 3 are not in order.

$1, 2, 3, 4$ has no inverseion

$4, 3, 2, 1$ has $\binom{4}{2}$ inversions

Inversions
The number of inversions of a permutation f is denoted by $\mathrm{inv}(f)$.

A permutation f is **even** if $\mathrm{inv}(f)$ is even; and is **odd** if $\mathrm{inv}(f)$ is odd.

Examples.

Transpositions

A transposition is a permutation of the form (i, j); that is, it swaps i and j and keeps everything else fixed.

Examples

Every permutation is a product of transpositions. Why?

Parity
The identity map is an even permutation.

A transposition is an odd permutation.

Let f be a permutation and τ a transposition.
Then if f is even, τf is odd
If f is odd, then τf is even.

Consequently; If f is a product of an even number of transpositions, f is even; and if f is a product of an odd number of transposition, then f is odd.

Consequently, we have the following chart for permutations:

$\circ$	even	odd
even	even	odd
odd	odd	even