Permutation Ladders

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1 Introduction

On a recent trip to South Korea the third author was re-introduced to a simple game which is prevalent in Asia. He first become acquainted with the game as a graduate student where the game was used to assign T.A's to exam problems for grading in a large lecture calculus section. During the authors' visit to South Korea the game was played each lunch time to determine who would pay for certain dishes. In this note we show how the game provides a simple pictorial view of permutations, and how it can be used to facilitate simple proofs of some of basic facts about permutations.

Here is a description of the game. A number of people, say 4, go to a restaurant for lunch. It is decided that each person will select one dish to be shared with the others. To make the lunch more interesting the game is played to decide who pays for each dish.

Let's assume that the people attending lunch are A, B, C and D; and that the dishes chosen are Kimchi (Korean cabbage), Bulgogi (Korean BBQ beef), Galbi (Korean ribs), and Pajeon (Korean seafood omelet), respectively. First,

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as illustrated in Figure 1, player A writes down each persons' name, in some order, at the top of a sheet of paper, places the name of their chosen dish at the bottom of the sheet of paper, draws vertical lines joining each name with the chosen dish, and finally covers up the names and dishes. Next each of B, C, and D draws several horizontal lines between adjacent vertical lines with the restriction that no two horizontal lines are at the same height. We refer to such a resulting diagram as a *ladder diagram*. For example, Figure 2 illustrates a ladder diagram.

Once the diagram is completed the players' names and dishes are revealed, and the assignment of dish to person is determined as follows. A person starts at the top of the diagram at his or her name and moves down a vertical line until he or she encounters a horizontal line (or a *rung*) incident to the vertical line. Each time a rung is encountered, the player follows the rung to the adjacent vertical line. The player repeats this process of moving down a vertical line, encountering a horizontal line, and following the rung to an adjacent line, until he or she arrives at one of the names of the dishes. This is the dish that the player must buy. For example, the Figure 3 indicates the path that person B follows. Person B must buy the Kimchi (which in this case means B is fortunate, since Kimchi is the least expensive dish selected).

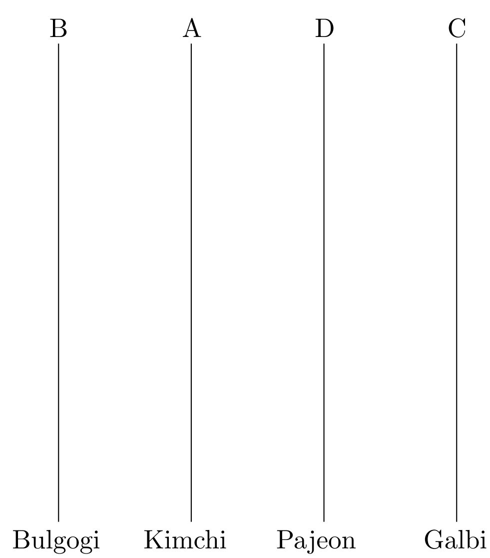

Figure 1

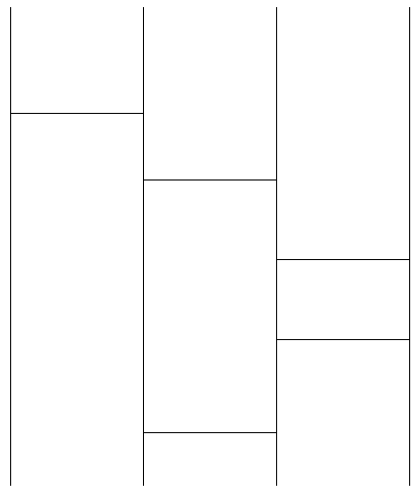

Figure 2

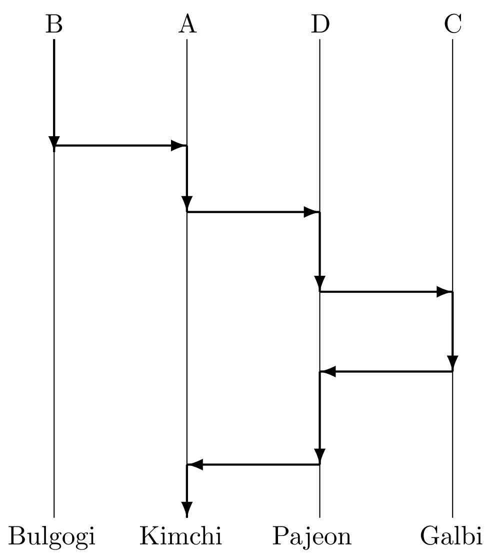

Figure 3

The other results from the game for Figure 3 are that A buys the Bulgogi, C buys the Galbi and D buys the Pajeon.

Let L be a ladder diagram with top row labelled by the elements of a finite set X, and bottom row labelled by the elements of a set Y with the same number of elements as X. Then the algorithm described above determines a function $f_L : X \to Y$. If M is a ladder diagram with top row labelled by the elements of Y (in the same order as they appear in L), and bottom row labelled by elements in a set Z with the same number of elements as Y, then we define $M \diamond L$ to be the ladder diagram obtained from L and M by identifying the bottom row of L with the top row of M. This is illustrated in Figure 4.

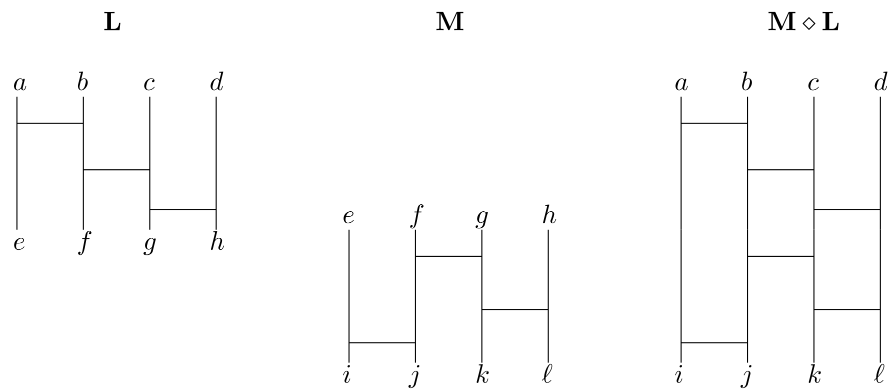

Figure 4

It is clear that $f_{M\diamond L} = f_M \circ f_L : X \to Z$, that is, $M \diamond L$ determines the function which is the composition of the functions f_M and f_L.

Clearly a ladder diagram with at most one rung is a bijection. Thus a ladder diagram with r rungs is the composition of r bijections. Since the composition of bijections is a bijection, we have the following:

Lemma 1.1 *Let L be a ladder diagram. Then f_L is a bijection.*

For the remainder of this note we will assume that the labels on the top and bottom of the ladder diagram are (from left to right) 1, 2, ..., n. In this setting, a ladder diagram L determines a permutation

$$\pi_L : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}.$$

In particular, the ladder diagram with no rungs determines the identity permutation, and the ladder diagram with exactly one rung, say the rung joining the ith and $(i+1)$st vertical line, determines the permutation g_i which interchanges i and $i + 1$ and keeps all other elements fixed.

In the next section we use ladder diagrams to give simple proofs of some fundamental facts about permutations. First we show that every permutation is the product of consecutive transpositions. Next we show that if L and M are ladder diagrams with $\pi_L = \pi_M$ then the number of rungs in L and the number of rungs in M have the same parity. From this we conclude that the notions of odd and even permutations are well-defined. Finally, for each permutation π we determine the least number of rungs in a ladder diagram L for which $\pi_L = \pi$.

2 Elementary Proofs

Let S_n denote the set of permutations on the set $\{1, 2, \ldots, n\}$. A *transposition* is a permutation σ for which there exist integers i and j with $i < j$ such that $\sigma(i) = j$, $\sigma(j) = i$ and $\sigma(k) = k$ for all $k \notin \{i, j\}$. If $j = i + 1$, then σ is a *consecutive* transposition. We now prove the well-known fact (see [**?**] for example) that every element of S_n is a product of consecutive transpositions. Since the ladder diagrams with one rung determine consecutive transpositions, this fact is equivalent to the fact that for each $\pi \in S_n$ there is a ladder diagram L for which $\pi_L = \pi$. If s is a positive integer and $s \leq n - 1$, we use s—$(s + 1)$ to denote the rung joining the sth and $(s + 1)$st vertical lines.

Theorem 2.1 *Let $\pi : X \to X$ be a bijection. Then there is a ladder diagram L such that $\pi_L = \pi$. In otherwords, every element in S_n is a product of consecutive transpositions.*

Proof. The proof is by induction on $|X|$. The result is clear if $|X| = 1$. Assume that $|X| > 1$ and proceed by induction. Let $j = \pi^{-1}(n)$. Let M be the ladder diagram whose rungs (from top to bottom) are

$$j\text{—}(j+1),\ (j+1)\text{—}(j+2),\ \ldots,\ (n-1)\text{—}n.$$

Let $\sigma = \pi_M$.

Note that $\sigma(j) = n$ and that $\pi \circ \sigma^{-1}$ is a bijection from X to X with $\pi \circ \sigma^{-1}(n) = n$. Let τ denote the permutation of $X \setminus \{n\}$ obtained by restricting $\pi \circ \sigma^{-1}$ to the set $X \setminus \{n\}$. By the inductive hypothesis there exists a ladder diagram K with $\pi_K = \tau$. Extending this ladder diagram by adding a vertical line on the right, we get a diagram K' with $\pi_{K'} = \pi \circ \sigma^{-1}$. Observe that the ladder diagram $K' \diamond M$ determines the function $(\pi \circ \sigma^{-1}) \circ \sigma = \pi$. The proof is now complete by induction. ■

The above result shows that if π is a permutation then π is a product of transpositions. It is a fundamental fact (see [**?**] for example) that if π is a product of k transpositions and also a product of ℓ transpositions, then k and ℓ both have the same parity, that is, either both k and ℓ are odd, or both k and ℓ are even. We now present a ladder diagram proof of this fact.

We first note that if σ is the transposition switching i and j where $i < j$, then $\pi_{L_\sigma} = \sigma$ where L_σ is the ladder diagram whose rungs from top to bottom are

$$i\text{—}i+1,\ (i+1)\text{—}(i+2),\ \ldots,\ (j-1)\text{—}j,\ (j-2)\text{—}(j-1),\ (j-3)\text{—}(j-2),\ \ldots,\ i\text{—}(i+1).$$

For example, Figure 5 illustrates L for the case that $n = 4$, $i = 1$ and $j = 4$.

Thus, if $\pi = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_k$ where σ_ℓ is the transposition switching i_ℓ and j_ℓ with $i_\ell < j_\ell$ (for $\ell = 1, 2, \ldots, k$), then the ladder diagram $L_{\sigma_1} \diamond L_{\sigma_2} \diamond \cdots \diamond L_{\sigma_k}$ determines π and has

$$\sum_{\ell=1}^{k}(2(j_\ell - i_\ell) - 1)$$

rungs. But

$$\sum_{i=1}^{k}(2(j_\ell - i_\ell) - 1)$$

is the sum of k odd integers and hence has the same parity as k. Thus to prove the fundamental fact it suffices to prove that the number of rungs in any two ladder diagrams which represent the same permutation have the same parity. This is done in the next theorem. First we have the following useful lemma. If L is a ladder diagram, then we let p_i denote the path traced out when one determines $\pi_L(i)$. For example, the path indicated in Figure 3 is p_1. We let $r(i, j)$ equal the number of rungs common to p_i and p_j.

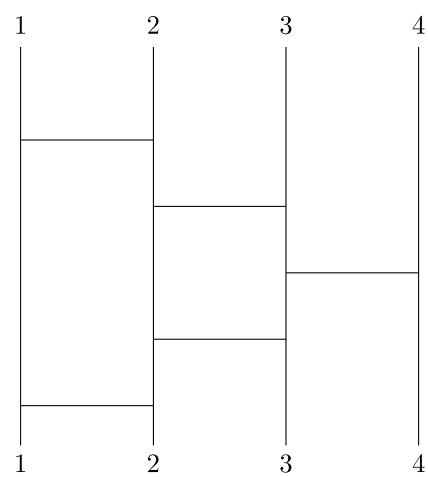

Figure 5

Lemma 2.2 *Let L be a ladder diagram representing π, and let i and j be integers with $i < j$. Then the number of rungs in common to the paths p_i and p_j is odd if $\pi(i) > \pi(j)$, and is even if $\pi(i) < \pi(j)$.*

Proof. Picture two particles α_i and α_j moving along the paths p_i and p_j, respectively, in such a way that at any time the particles are at the same height. Initially, α_i starts to the left of α_j. Since the particles are always at the same height, the only way that the particle on the left will move past the particle on the right is when the two particles collide. The collisions occur at and only on a rung which in common to both p_i and p_j, and there is exactly one collision per such rung. Hence, if k is the number of rungs common to p_i and p_j, then the particles move past each other exactly k times. If $\pi(i) < \pi(j)$, then particle α_i starts and ends on the left of particle α_j, and hence k is even. If $\pi(i) > \pi(j)$, then α_i starts on the left of α_j, but ends on the right. Hence, if $\pi(i) > \pi(j)$, k must be odd. ∎

Given a ladder diagram L whose rungs from top to bottom are $r_1, r_2, \ldots, r_m$, we let L^{-1} denote the ladder diagram whose rungs from top to bottom are $r_m, r_{m-1}, \ldots, r_1$. It is clear that $\pi_{L^{-1}}$ is the inverse of the permutation π_L.

Theorem 2.3 *Let L_1 and L_2 be ladder diagrams which represent the permutation π. Then the number of rungs in L_1 and the number of rungs in L_2 have the same parity.*

Proof. We first prove the result in the case that π is the identity map. Let L be a ladder diagram that represents π. Now suppose i and j are integers with $1 \leq i < j \leq n$. Since the path p_i goes from i to i, and the path p_j goes from j to j, Lemma **??** implies that $r(i,j)$ is even. Thus, $\sum_{i<j} r(i,j)$ is even. The fact that π_L is a bijection implies that each rung of the ladder is in exactly two paths. For if p_i, p_j and p_k all have a rung r is common, then two of these paths will coalesce at this rung, and hence will terminate at the same point. Thus $\sum_{i<j} r(i,j)$ is equal to the number of rungs in L. Therefore, L has an even number of rungs.

Now we prove the more general result. Let π be an arbitrary permutation. If L_1 and L_2 are ladder diagrams for π, then $L_2^{-1} \diamond L_1$ is a ladder diagram for the identity map. Hence by the above argument $L_2^{-1} \diamond L_1$ has an even number of rungs. But clearly the number of rungs of $L_2^{-1} \diamond L_1$ is the sum of the number of rungs in L_1 and in L_2. Hence, the number of rungs in L_1 and in L_2 have the same parity. ∎

A permutation is *even* if it is a product of an even number of transpositions, and *odd* if it is a product of an odd number of transpositions. Theorem **??** shows that a permutation is either even or odd, and no permutation is both even and odd.

Given a permutation π let $\ell(\pi)$ denote the least number of consecutive transpositions whose product is π. Thus $\ell(\pi)$ equals the least number of rungs in a ladder diagram that represents π. An *inversion* of π is an ordered pair of integers (i, j) such that $i < j$ and $\pi(i) > \pi(j)$. Let $\text{inv}(\pi)$ denote the number of inversions of π. The proof of the following result utilizes ladder diagrams.

Theorem 2.4 *Let π be a permutation in S_n. Then*

$$\ell(\pi) = inv(\pi).$$

Proof. Let L be a ladder diagram with $\pi_L = \pi$. Let $i < j$. By Lemma **??**, $r(i, j)$ is odd if (i, j) is an inversion, and $r(i, j)$ is even if (i, j) is not an inversion. Since the number of rungs equals $\sum_{i<j} r(i, j)$, each inversion contributes at least one to the sum and it follows that L has at least $\text{inv}(\pi)$ rungs. Hence $\ell(\pi) \geq \text{inv}(\pi)$.

We now describe a way to construct a ladder diagram which represents π and has exactly $\text{inv}(\pi)$ rungs. Consider the n-tuple

$$s_1 = (\pi(1), \pi(2), \ldots, \pi(n)).$$

From this we construct a finite sequence $s_1, s_2, \ldots, s_m$ of n-tuples as follows. Suppose $s_1, s_2, \ldots, s_k$ have been defined. If $s_k = (1, 2, \ldots, n)$ we stop and set $m = k$. Otherwise, let i be the smallest positive integer such that the ith entry of s_k is not i. We obtain s_{k+1} from s_k by interchanging the entry i and the entry in the position immediately to its left. For example, the sequence if $s_1 = (3, 2, 4, 1)$ is given in Figure 6. Intuitively, $s_1, s_2, \ldots, s_m$ is the sequence of n-tuples obtained from s_1 by moving 1 to the first position, then moving 2 to the second position, etc.

For $t = 1, 2, \ldots, m-1$, the sequences s_t and s_{t+1} determine a rung, r_t. Namely, if s_{t+1} is obtained from s_t by interchanging the entries in positions p and $p+1$ then r_t is the rung joining p and $p+1$.

Let M be the ladder diagram whose rungs from top to bottom are the rungs $r_1, r_2, \ldots, r_{m-1}$. The ladder diagram M in the case that $s_1 = (3, 2, 4, 1)$ is illustrated in Figure 6.

Just as the ith entry of s_1 equals the $\pi(i)$th entry of s_m, the path starting at the ith vertical line terminates at the $\pi(i)$th vertical line, and hence M represents π. Thus, as argued above $m - 1 \geq \text{inv}(\pi)$.

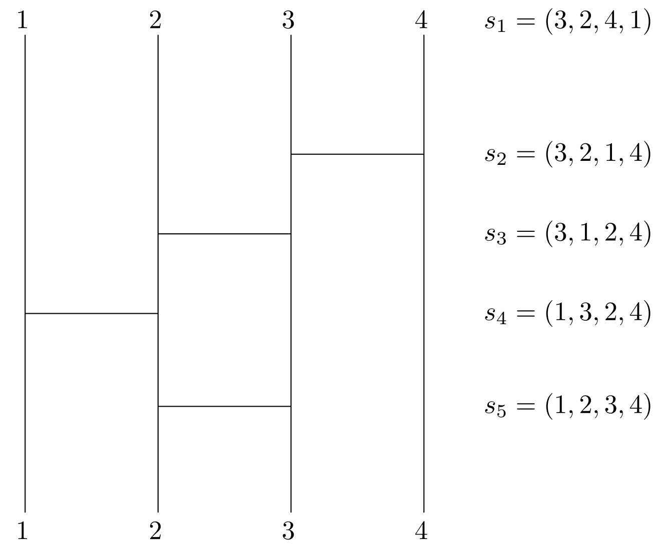

Figure 6

We now show that each rung of M determines a different inversion of π. Namely, for $t = 1, 2, \ldots, m-1$, let u and v with $u > v$ be the 2 symbols that are interchanged in s_t to get s_{t+1}. Let i and j be the integers with $u = \pi(i)$ and $v = \pi(j)$. By the way that $s_1, s_2, \ldots, s_m$ are constructed, $i < j$ and $\pi(i) > \pi(j)$. Hence (i, j) is an inversion. Since in the construction smaller elements move only to the left, distinct rungs give rise to different inversions. Thus, $m - 1 \leq \text{inv}(\pi)$.

Therefore, M is a ladder diagram with $\pi_M = \pi$ and exactly $\text{inv}(\pi)$ rungs. ■

3 Conclusion

It has been our experience that the ladder diagram approach to permutations not only grabs our students' attention, but also facilitates their understanding of permutations. Our exam results show that many of the students can internalize the ladder diagram approach and can use it to justify why some permutations are even and some are odd.

In addition, the ladder diagram approach often invokes students to ask some interesting questions (which we leave for the reader to ponder) such as:

(a) Given a ladder diagram how can one quickly determine the permutation it corresponds to?

(b) Given a ladder diagram L, how can one efficiently determine whether or not L has the fewest number of rungs for permutation it represents?

(c) Given a permutation π, how many different ladder diagrams L with $\text{inv}(\pi)$ rungs represent π?

(d) What happens if we identify the top and bottom of the ladder, in particular how many "loops" will we get?

We do not advocate the exclusive use of the ladder diagram approach. In our class we introduce permutations as special types of functions, describe the two-line notation for permutations, use ladder diagrams to discuss inverses and signs of permutations, and then discuss cycle notation.

Children in Korea often play the ladder game to decide who will be "it". We refer the reader [**?**] to a delightful survey of other ways that children in different cultures choose who will be "it". The book [**?**] describes childrens' games from around the world, and [**?**] is an excellent reference for those interested in other Korean games.

We hope that this note has successfully demonstrated the ancient Korean adage

(Translation: One sight is better than one hundred sayings) and the equivalent Western saying "A picture is worth a thousand words".

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