MathTLC Algebra & Number Theory
Summer 2009
Homework # 3: Due Tuesday, July 14

1. Simplify the following products of cycles:
   (a) (135)(53)(34)(3256781)
   (b) $(1567432)^{21242}$
   (c) $(134789)^{-1}$
   (d) (15)(14)(13)(12)
   (e) $(1325)(27531)(1325)^{-1}$

2. Let G be the set consisting of the 4 elements a, b, c, and d. Define an operation on G by using the following table:

$*$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	d
d	d	c	d	a

   (a) Show that G has an identity.
   (b) Show that for each element $x \in G$, the is an element $y \in G$ such that $x * y = y * x = e$. Note that from the table we have that $d * a = d$ and $d * c = d$. Explain why this implies that $*$ is not associative.
   (c) Give a particular example that shows that $*$ is not an associative binary operation.

3. ("Do the shuffle" problem)
   This problem concerns perfect shuffles of a deck of 16 cards.
   (a) Find the permutation, ℓ corresponding to a left shuffle.
   (b) Find the permutation r corresponding to a right shuffle.
   (c) Find the permutation corresponding to a left shuffle followed by a right shuffle by "physically doing the shuffle", and then by computing $r \circ \ell$.
   (d) What would the original deck look like after you did 5 left shuffles?
   (e) What would the original deck look like after you did a left, right, left, right, left, right, left, right shuffle?
   (f) What kind of permutation is ℓ?
   (g) What kind of permutation if r?
   (h) Could I start with the original deck, do a bunch of left or right shuffles in some order, and end up at with the cards in the order $2, 3, 4, 5, 1, 7, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16$? Why or why not?

4. ("The 15 puzzle problem")
The problem concerns the 15-puzzle. For a description see.

   (a) Any configuration of the 15-puzzle corresponds to a permutation of 1,2,..., 16, where we take 16 to be the blank tile. Find the permutation, f, (in cycle notation) for

4	3	2	1
5	6	7	8
12	11	10	9
16	15	14	13

   (b) A basic move consists of sliding one of the tiles into a blank space. For example, in the above configuration we can slide the tile labelled 12 down, or the tile labeled 15 to the right. For each of these two moves find a transposition τ so that τf is the permutation corresponding to the configuration after you did one basic move.

   (c) Explain why each basic move corresponds to a transposition.

   (d) Explain why each sequence of moves that starts with the correct configuration

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

   and ends with a configuration which has the blank tile in the lower right corresponds to an even permutation. (Hints: Why does this require an even number of basic moves?, and What goes up must come down).

   (e) Give an example of a configuration of tiles that can't be moved to the correct configuration, and provide a detailed argument why this can't be done.

5. (Rigid motions of the hexagon)

   (a) Describe all the rigid motions of a regular hexagon.

   (b) Determine the permutation corresponding to each rigid motion in (a). (Here we will label the vertex in the upper left with 1, and the label the others as 2,3,4,5,6 in a clockwise fashion).

   (c) Which of the rigid motions give even permutations?

   (d) Which rigid motions r have the propery that $r \circ s = s \circ r$ for every rigid motion s of the hexagon?

6. (Rigid motions of the cube)
Let G be the group of rigid motions of the cube.

(a) Explain why G has at most 24 elements.

(b) Describe geometrically a rigid motion f such that $f^2 = \mathrm{id}$ but $f \neq \mathrm{id}$. Write down the permutation of the vertices corresponding to f.

(c) Describe geometrically a rigid motion g such that g^4 is the identity, but none of g, g^2 or g^3 is the identity. Write down the permutation of the vertices corresponding to g.

(d) Describe geometrically a rigid motion h such that h^3 is the identity, but none of h, h^2 is the identity. Write down the permutation of the vertices corresponding to h.