

## Inversions

Recall:  $(i, j)$  is an inversion of the permutation  $f$  provided  $i < j$  but  $f(i) > f(j)$ .

Note that an inversion of  $f$  corresponds to to entries in  $f(1), f(2), \dots, f(n)$  that are not in increasing order.

Example:

$f(1) = 4, f(2) = 3, f(3) = 1, f(4) = 2$

4,3,1,2 gives us 5 inversions

So  $\text{inv}(f) = 5$ .

Inversions  $\text{inv}(\text{id}) = 0$

$\text{inv}(r) = \binom{n}{2}$ , where  $r$  is the permutation of  $1, 2, \dots, n$  that send  $i$  to  $n - i$

Question: How many inversions does a left shuffle of  $n$  cards have?

Question: How many inversions does a right shuffle of  $n$  card have?

Transpositions A **transposition** is a permutation of the form  $(i, j)$ ; that is a transposition just switches two elements.

$f(1) = 1; f(2) = 3; f(3) = 2; f(4) = 4$  is the transposition  $(2, 3)$ .

If  $\tau$  is a transposition, then  $\tau$  has exactly one inversion

Even & Odd Permutations A permutation  $f$  is **odd** if it has an odd number of inversions.

A permutation  $f$  is **even** if it has an even number of inversions.

Examples:

The identity is an even permutation

Each transposition is an odd permutation

The permutation corresponding to 4, 5, 1, 2, 3 is an even permutation

Even & Odd Permutations Let's investigate how the evenness/oddness of a permutation  $f$  is affected by multiplying it by a transposition  $\tau$ .

Example:  $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 2 & 9 & 1 & 7 & 6 & 9 & 8 \end{pmatrix}$

$\tau = (2, 6)$ .

Then  $\tau \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & \mathbf{6} & 9 & 1 & 7 & \mathbf{2} & 9 & 8 \end{pmatrix}$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 2 & 9 & 1 & 7 & 6 & 9 & 8 \end{pmatrix}, \quad \tau = (2, 6).$$

$$\tau \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & \mathbf{6} & 9 & 1 & 7 & \mathbf{2} & 9 & 8 \end{pmatrix}$$

For the most part an inversion of  $f$  gives an inversion of  $\tau$  and vice versa.

The only changes are the pairs involving 6, 2, or both

Note for each entry  $j$  between the 6 and 2, we either get an inversion  $j6$  in  $f$  or an inversion  $6j$  in  $\tau f$  but not both; and either an inversion  $2j$  in  $f$  or an inversion  $j2$  in  $\tau f$ .

So

$$\text{inv}(f) + \text{inv}(\tau f) = 2 \# \text{inversions not involving } 6 \text{ and } 2 + 2 \cdot 3 + 1.$$

So  $\text{inv}(f) + \text{inv}(\tau f)$  is odd.

Upshot: **Multiplication by a transposition changes the parity.**

if  $f$  is odd,  $\tau f$  is even;

if  $f$  is even  $\tau f$  is odd.

Every permutation is a product of transpositions!

Example:

$$(1, 3, 4, 5, 6, 8) = (1, 8)(1, 6)(1, 5)(1, 4)(1, 3)$$

$$(1, 3, 4, 5, 6, 8)(2, 7, 9, 10) = \\ (1, 8)(1, 5)(1, 4)(1, 3)(2, 10)(2, 9)(2, 10)$$

$$\text{id} = (1, 2)(1, 2)$$

## Consequences:

- A permutation is odd if and only if it is a product of an odd number of transpositions.
- A permutation is even if and only if it is a product of an even number of transpositions.
- An odd cycle is an even permutation
- An even cycle is an odd permutation.
- An even permutation composed with an even permutation is even.
- An even permutation composed with an odd permutation is odd.
- An odd permutation composed with an odd permutation is even.

Train track problem You have an oval train track; 10 cars in order 1,2,3,..., 11; a round about that allows you to reverse the order of a section of 4 trains; and you can move the trains around the oval.

Can you ever get the trains to be in the order 1, 2, 4, 3, 5, 6, 7, 9, 10, 8, 11?

Valid operations correspond to  $f = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)$  which is an even permutation, and  $g = (1, 4)(2, 3)$  which is an even permutation. So any product of  $f$ 's and  $g$  will be even.

The final configuration you want corresponds to the permutation  $(3, 4)(8, 10, 9)$  which is an odd permutation.

So it is impossible!

## Perfect Shuffles of a deck of 16 cards

- 1 Find the permutation,  $\ell$  corresponding to a left shuffle.
- 2 Find the permutation  $r$  corresponding to a right shuffle.
- 3 Find the permutation corresponding to a left shuffle followed by a right shuffle by “physically doing the shuffle”, and then by computing  $r \circ \ell$ .
- 4 What would the original deck look like after you did 5 left shuffles?
- 5 What would the original deck look like after you did a left, right, left, right, left, right, left, right shuffle?
- 6 What kind of permutation is  $\ell$ ?
- 7 What kind of permutation if  $r$ ?
- 8 Could I start with the original deck, do a bunch of left or right shuffles in some order, and end up at with the cards in the order 2, 3, 4, 5, 1, 7, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16? Why or why not?

## 15 puzzle

- Description
- Every basic move corresponds to a transposition
- Every sequence of moves that starts and ends with a blank in the lower right involves an even number of moves
- Every sequence of moves corresponds to an even permutation
- Upshot: No odd permutation of the cells 1-15 can be put into the correct order.