Even & Odd Permutations

Let's investigate how the evenness/oddness of a permutation f is affected by multiplying it by a transposition τ.

Example: $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 2 & 9 & 1 & 7 & 6 & 8 & 5 \end{pmatrix}$

$\tau = (2, 6)$.

Then $\tau \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & \mathbf{6} & 9 & 1 & 7 & \mathbf{2} & 8 & 5 \end{pmatrix}$

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$$\tau \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & \mathbf{6} & 9 & 1 & 7 & \mathbf{2} & 8 & 5 \end{pmatrix}$$

Let's play matchmaker and try to pair the inversions of f with those of $\tau \circ f$.

Can pair any inversion of f not involving 3rd and 7th column with the same inversion of τf.

Examples:

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 2 & 9 & 1 & 7 & 6 & 8 & 5 \end{pmatrix},$$

$$\tau \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 6 & 9 & 1 & 7 & 2 & 8 & 5 \end{pmatrix}$$

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Can pair any inversion of f involving column 3 and a column not in $\{3, 4, 5, 6, 7\}$ witth column 7 and the same column of τf

Examples:

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & \color{blue}{3} & \color{blue}{2} & 9 & 1 & 7 & 6 & 8 & 5 \end{pmatrix},$$

$$\tau \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & \color{blue}{3} & 6 & 9 & 1 & 7 & \color{blue}{2} & 8 & 5 \end{pmatrix}$$

Can pair any inversion of f involving column 7 and a column not in $\{3, 4, 5, 6, 7\}$ witth column 3 and the same column of τf

Examples:

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 2 & 9 & 1 & 7 & \color{blue}{6} & 8 & \color{blue}{5} \end{pmatrix},$$

$$\tau \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & \color{blue}{6} & 9 & 1 & 7 & 2 & 8 & \color{blue}{5} \end{pmatrix}$$

For $j \in \{4, 5, 6\}$:

Exactly one of the following holds: $3\&j$ is an inversion in f
$j\&7$ is an inversion in τf
Exactly one of the following holds: $j\&7$ is an inversion in f
$3\&j$ is an inversion of τ.

Upshot:
The total number of inversions in f and τf involving column 3 or column 7, and a column in $\{4, 5, 6\}$ is $2 \cdot 3$.

Note that 3&7 is an inversion of exactly one of f and τf.

So we have that $\text{inv}(f) + \text{inv}(\tau f)$ equals
1
$+2 \cdot 3$
$+2 *$ ($\#$of inv of f not involving col 3 or col 7 & a col in $\{4, 5, 6\}$

Upshot:
$\text{inv}(f) + \text{inv}(\tau f)$ is odd

Multiplication by a transposition changes the parity.

if f is odd, τf is even;

if f is even τf is odd.

Example:

$(1,3,4,5,6,8) = (1,8)(1,6)(1,5)(1,4)(1,3)$

$(1,3,4,5,6,8)(2,7,9,10) =$
$(1,8)(1,5)(1,4)(1,3)(2,10)(2,9)(2,10)$

$\mathrm{id} = (1,2)(1,2)$

Consequences:

- A permutation is odd if and only if it is a product of an odd number of transpositions.
- A permutation is even if and only if it is a product of an even number of transpositions.
- An odd cycle is an even permutation
- An even cycle is an odd permutation.
- An even permutation composed with an even permutation is even.
- An even permutation composed with an odd permutation is odd.
- An odd permutation composed with an odd permutation is even.

Train track problem

You have an oval train track; 10 cars in order 1,2,3,$\ldots$, 11; a round about that allows you to reverse the order of a section of 4 trains; and you can move the trains around the oval.

Can you ever get the trains to be in the order $1, 2, 4, 3, 5, 6, 7, 9, 10, 8, 11$?

Valid operations correspond to $f = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)$ which is an even permutation,
and $g = (1, 4)(2, 3)$ which is an even permutation. So any product of f's and g will be even.

The final configuration you want corresponds to the permutation $(3, 4)(8, 10, 9)$ which is an odd permutation.

So it is impossible!

Perfect Shuffles of a deck of 16 cards

1. Find the permutation, ℓ corresponding to a left shuffle.
2. Find the permutation r corresponding to a right shuffle.
3. Find the permutation corresponding to a left shuffle followed by a right shuffle by "physically doing the shuffle", and then by computing $r \circ \ell$.
4. What would the original deck look like after you did 5 left shuffles?
5. What would the original deck look like after you did a left, right, left, right, left, right, left, right shuffle?
6. What kind of permutation is ℓ?
7. What kind of permutation if r?
8. Could I start with the original deck, do a bunch of left or right shuffles in some order, and end up at with the cards in the order $2, 3, 4, 5, 1, 7, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16$? Why or why not?

15 puzzle

- Description
- Every basic move corresponds to a transposition
- Every sequence of moves that starts and ends with a blank in the lower right involves an even number of moves
- Every sequence of moves corresponds to an even permutation. Why? Why is it a product of an even number of transpositions?
- Upshot: No odd permutation of the cells 1-15 can be put into the correct order.

A **permutation group** is a subset G of permutations s.t.
(a) id $\in G$
(b) G is closed under composition
(c) G is closed under inverses.

Examples:

All the permutations of $\{1, 2, \ldots, n\}$. Denoted by S_n and called the symmetric groups

All the even permutations of $\{1, 2, \ldots, n\}$. Denote by A_n and called the Alternating group.

The rigid motions of a figure give permutation groups of vertices, edges, faces

The Rubik's cube gives a permutation group of the cells of the group

Rigid Motions of Square

Rigid motions of a square-pinwheel

Rigid motions of cube