Spectrally and inertially arbitrary sign patterns

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Received 18 September 2003; accepted 2 June 2004
Submitted by R.A. Brualdi

Abstract

We introduce some $n$-by-$n$ sign patterns which allow for arbitrary spectrum and hence also arbitrary inertia. Consequently, we demonstrate that some known inertially arbitrary patterns are in fact spectrally arbitrary. We demonstrate that all inertially arbitrary patterns of order 3 are spectrally arbitrary and classify all spectrally arbitrary patterns of order 3. We illustrate that in general, the class of spectrally arbitrary patterns is distinct from the inertially arbitrary patterns, and present some observations about inertially arbitrary patterns.

AMS classification: 15A18; 05C20

Keywords: Sign pattern; Inertia; Spectrum; Nilpotent; Potentially stable

1. Introduction

An $n \times n$ sign pattern is a matrix $S$ with entries in $\{+, -, 0\}$. The set of all real matrices with the same sign pattern as $S$ is the qualitative class

$$Q(S) = \{ A \in M_n(R) : \text{sign}(a_{ij}) = s_{ij} \text{ for all } i, j \}.$$ 

The spectrum of a sign pattern $S$ is the collection of all multisets $U$ of $n$ complex numbers such that $U$ consists of the eigenvalues of some matrix $A \in Q(S)$. A sign pattern $S$ is a spectrally arbitrary pattern (SAP) if every multiset of $n$ complex
numbers, closed under complex conjugation, is in the spectrum of $S$. Spectrally arbitrary patterns must necessarily be both potentially stable and potentially nilpotent.

A couple of recent papers [1,8] introduce some sign patterns which are spectrally arbitrary for all orders $n \geq 2$. In this paper, we introduce some other sign patterns, which are spectrally arbitrary for all orders $n \geq 2$. Much of this work is motivated by the paper [2] by Drew et al. where the tridiagonal sign pattern

$$T_n = \begin{bmatrix}
- & + & 0 & \cdots & \cdots & 0 \\
- & 0 & + & & & \\
0 & - & 0 & + & \ddots & \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 0 & - & +
\end{bmatrix}$$

is introduced. $\mathcal{F}_n$ is spectrally arbitrary for $2 \leq n \leq 16$ (see [2–4]), and is conjectured to be spectrally arbitrary for all $n$. In Section 2, we define a class of sparse sign patterns $\mathcal{D}_{n,r}$ which contains the pattern $\mathcal{F}_n$; namely $\mathcal{D}_{n,2} = \mathcal{F}_n$. While we do not solve the conjecture, we do show that other patterns in the class $\mathcal{D}_{n,r}$ are spectrally arbitrary patterns, in fact minimally spectrally arbitrary. We also determine that not all the patterns $\mathcal{D}_{n,r}$ are SAPs.

The inertia of a matrix $A$ is an ordered triple $i(A) = (n_+(A), n_-(A), n_0(A))$ where $n_+(A)$ is the number of eigenvalues of $A$ with positive real part, $n_-(A)$ is the number of eigenvalues with negative real part, and $n_0(A)$ is the number of eigenvalues with zero real part. The inertia of a sign pattern $\mathcal{F}$ is $i(\mathcal{F}) = \{i(A) \mid A \in \mathbb{Q}(\mathcal{F})\}$. An $n$-by-$n$ sign pattern $\mathcal{F}$ is an inertially arbitrary pattern (IAP) if $i(\mathcal{F})$ contains every ordered triple $(n_1, n_2, n_3)$ with $n_1 + n_2 + n_3 = n$. If a sign pattern is spectrally arbitrary, it must also be inertially arbitrary.

For example, we will examine the $(2r - 1)$-diagonal sign pattern of order $n$, $\mathcal{F}_{n,r} = \begin{bmatrix}
- & + \cdots + & 0 & \cdots & \cdots & 0 \\
- & 0 & + & \ddots & & \\
\vdots & - & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 \cdots & \cdots & \cdots & 0 & - & + \\
\vdots & \ddots & \ddots & \ddots & \ddots & + \\
\vdots & \ddots & \ddots & \ddots & \ddots & + \\
0 \cdots & \cdots & 0 & - & \cdots & - +
\end{bmatrix}$

Gao and Shao [6] showed that $\mathcal{F}_{n,n}$ is inertially arbitrary whereas Miao and Li [9] demonstrated that $\mathcal{F}_{n,n-1}$ is inertially arbitrary. Britz et al. [1] demonstrate that
$S_{n,n}$ is in fact spectrally arbitrary. In Section 3 we determine that $S_{n,r}$ is spectrally arbitrary whenever $n \leq 2r$.

In Section 4 we generalize the class of patterns $D_{n,r}$ to construct more spectrally arbitrary patterns.

A pattern $\mathcal{F}$ is signature similar to pattern $\mathcal{S}$ if $\mathcal{F} = \mathcal{P} \mathcal{S} \mathcal{P}^T$ where $\mathcal{P}$ is a diagonal matrix with diagonal entries from $\{+, -\}$. We will refer to a signature transformation by the entries on its main diagonal. If $\mathcal{S}$ is spectrally or inertially arbitrary, then so is any matrix obtained from $\mathcal{S}$ via a signature similarity. Likewise if $\mathcal{S}$ is spectrally or inertially arbitrary, then so is $-\mathcal{S}$ or any pattern obtained from $\mathcal{S}$ via transposition, or permutation similarity. We say a sign pattern $\mathcal{F}$ is equivalent to $\mathcal{S}$ if $\mathcal{S}$ can be obtained from $\mathcal{F}$ by a sequence of the four transformations just mentioned. In Section 5, we classify, up to equivalence, the sign patterns of order 3 which are IAPs (likewise SAPs) and present some further observations about inertially arbitrary patterns.

2. A sparse sign pattern

We begin by considering the $n$-by-$n$ sign pattern

$$D_{n,r} = \begin{bmatrix}
- & + & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
- & 0 & + & 0 & \cdots & \cdots & \cdots & \\
- & 0 & 0 & + & \cdots & \cdots & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \\
- & 0 & \cdots & \cdots & 0 & 0 & \cdots & \\
0 & \cdots & 0 & \cdots & 0 & + & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & \cdots & 0 & \cdots & 0 & + & \cdots & \\
\end{bmatrix}$$

with $r$ negative entries in the first column, $2 \leq r \leq n$. Note that $D_{n,2} = T_n$. We show that $D_{n,r}$ is spectrally arbitrary for $n \leq 2r$. Let

$$A_{n,r} = \begin{bmatrix}
-a_1 & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
-a_2 & 0 & 1 & 0 & \cdots & \cdots & \cdots & \\
-a_3 & 0 & 0 & 1 & \cdots & \cdots & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \\
-a_r & 0 & \cdots & \cdots & 0 & 0 & \cdots & \\
0 & -a_{r+1} & 0 & \cdots & \cdots & 1 & 0 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & \cdots & 0 & -a_n & 0 & \cdots & 0 & 1 \\
\end{bmatrix}$$
be an \(n\)-by-\(n\) matrix with \(a_i > 0\) for all \(i\). Then \(A_{n,r} \in \mathcal{Q}(\mathcal{D}_{n,r})\). We use the notation \(A^{(n)}_{n,r}\) to denote the matrix obtained from \(A_{n,r}\) by changing the \((n,n)\) entry to 0. Observe that if \(r = n\) then \(A^{(n)}_{n,r}\) is simply the companion matrix

\[
C_n = \begin{bmatrix}
-a_1 & 1 & 0 & \cdots & 0 \\
-a_2 & 0 & 1 & \ddots & \\
-a_3 & 0 & 0 & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & 1 \\
-a_n & 0 & \cdots & 0 & 0
\end{bmatrix}
\]

which has characteristic polynomial \(\det(xI - C_n) = \sum_{i=0}^{n} a_i x^{n-i}\) if we set \(a_0 = 1\).

Note that while \(\mathcal{D}_{n,r}\) is defined for all \(n \geq 2\), the matrix \(C_n\) is defined for \(n \geq 1\). For convenience we also let \(\det(xI - C_0) = 1\).

Lemma 2.1. If \(n \leq 2r\) and \(a_0 = 1\), then the characteristic polynomial of \(A^{(n)}_{n,r}\) is

\[
\sum_{i=0}^{r-1} a_i x^{n-i} + \sum_{i=r}^{n} \left[ a_{i-r} \left( \sum_{k=i}^{n} a_k \right) x^{n-i} \right].
\]

Proof. Suppose \(r < n\). By expansion along the last row we note that if \(n \leq 2r\), then

\[
\det(xI - A^{(n)}_{n,r}) = a_n \det(xI - C_{n-r}) + x \det(xI - A^{(n-1)}_{n-1,r})
\]

since the cofactor of entry \(a_n\) is the determinant of a block lower-triangular matrix with two diagonal blocks: \(C_{n-r}\) and a triangular matrix with \(-1\)’s on the diagonal. Hence by induction on \(n\) we find the characteristic polynomial of \(A^{(n)}_{n,r}\) is

\[
a_n \sum_{i=0}^{n-r} a_i x^{n-r-i} + \sum_{i=r}^{n-1} a_i x^{n-i} + \sum_{i=r}^{n-1} \left[ a_{i-r} \left( \sum_{k=i}^{n-1} a_k \right) x^{n-i} \right]
\]

\[
= \sum_{i=0}^{r-1} a_i x^{n-i} + \sum_{i=r}^{n} \left[ a_{i-r} \left( \sum_{k=i}^{n} a_k \right) x^{n-i} \right]. \quad \Box
\]

Lemma 2.2. If \(n \leq 2r\), \(a_0 = 1\) and \(a_{-1} = 0\), then the characteristic polynomial of \(A_{n,r}\) is

\[
\sum_{i=0}^{r-1} (a_i - a_{i-1}) x^{n-i} + \sum_{i=r}^{n} a_i - a_{r-1} x^{n-r}
\]

\[
+ \sum_{i=r+1}^{n} \left[ a_{i-r} \left( \sum_{k=i}^{n} a_k \right) - a_{i-r-1} \left( \sum_{k=i-1}^{n-1} a_k \right) \right] x^{n-i}.
\]
Proof. If \( n \leq 2r \) then by expansion along the last row we have
\[
\det(x I - A_{n,r}) = a_n \det(x I - C_{n-r}) + (x - 1) \det(x I - A_{n-1,r}^{(n-1)})
\]
Thus using Lemma 2.1, we have
\[
\det(x I - A_{n,r}) = a_n \sum_{i=0}^{n-r} a_i x^{n-r-i} + \sum_{i=0}^{r-1} a_i x^{n-i} + \sum_{i=r}^{n-1} \left( \sum_{k=i}^{n-1} a_k \right) x^{n-i} - \sum_{i=0}^{r-1} a_i x^{n-i-1} - \sum_{i=r}^{n-1} \left( \sum_{k=i}^{n-1} a_k \right) x^{n-i-1}
\]
from which the formula follows. □

We use the method of Observation 10 in [2] in the following argument to demonstrate that \( D_{n,r} \) is spectrally arbitrary for \( n < 2r \). To show that a pattern \( S \) is spectrally arbitrary it is equivalent to show that every polynomial
\[
x^n + b_1 x^{n-1} + b_2 x^{n-2} + \cdots + b_{n-1} x + b_n
\]
with real coefficients \( b_i \) is the characteristic polynomial of some matrix \( A \in Q(S) \). Since we know the characteristic polynomial of \( A_{n,r} \) from Lemma 2.2, we will define the following functions of the variables \( a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \):
\[
fi = ai - ai-1 - bi \quad \text{for } 1 \leq i \leq r-1,
fr = \left( \sum_{i=r}^{n} ai - a_{r-1} \right) - br,
fi = ai-r \left( \sum_{k=i}^{n} a_k \right) - ai-r-1 \left( \sum_{k=i-1}^{n-1} a_k \right) - bi \quad \text{for } r+1 \leq i \leq n,
\]
where \( a_0 = 1 \). To demonstrate that \( D_{n,r} \) is spectrally arbitrary it is sufficient to show that given any \( b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n \), there are positive real numbers \( a_1, a_2, \ldots, a_n \) such that \( f_i = 0 \) for all \( i \).

Note that for every sign pattern \( \mathcal{S} \), \( A \in Q(\mathcal{S}) \) if and only if \( cA \in Q(\mathcal{S}) \) for every \( c > 0 \). Also if
\[
\det(x I - A) = x^n + b_1 x^{n-1} + b_2 x^{n-2} + \cdots + b_{n-1} x + b_n,
\]
then
\[
\det(x I - cA) = x^n + cb_1 x^{n-1} + c^2 b_2 x^{n-2} + \cdots + c^{n-1} b_{n-1} x + c^n b_n.
\]

Thus it suffices to show that given any \( b = (b_1, b_2, \ldots, b_n) \) arbitrarily close to \((0, 0, \ldots, 0)\), there is a positive vector \( a = (a_1, a_2, \ldots, a_n) \) such that \( f_i = 0 \) for all \( i \). We will use the Implicit Function Theorem.

Note that each of the functions \( f_i \) has continuous partial derivatives with respect to all \( 2n \) variables. The Jacobian
\[ J = \frac{\delta(f_1, f_2, \ldots, f_n)}{\delta(a_1, a_2, \ldots, a_n)} \]

is the determinant of the matrix with entries \((i, j)\) equal to \(\frac{\delta(f_i)}{\delta(a_j)}\). In this case, \(J\) is the determinant of a block lower triangular matrix

\[
\begin{bmatrix}
  W & 0 \\
  * & M
\end{bmatrix},
\]

where \(W\) itself is an \((r - 1)\)-by-\((r - 1)\) lower triangular matrix with ones on the main diagonal and

\[
M = \begin{bmatrix}
  1 & 1 & 1 & \cdots & 1 & 1 \\
  -1 & (a_1 - 1) & (a_1 - 1) & \cdots & (a_1 - 1) & a_1 \\
  0 & -a_1 & (a_2 - a_1) & \cdots & (a_2 - a_1) & a_2 \\
  0 & 0 & -a_2 & \ddots & \vdots & \vdots \\
  \vdots & \ddots & \ddots & \ddots & (a_{n-r-1} - a_{n-r-2}) & \vdots \\
  0 & \cdots & \cdots & \cdots & 0 & -a_{n-r-1} a_{n-r}
\end{bmatrix}
\]

if \(n < 2r\). By way of elementary row operations, \(M\) is row equivalent to an upper triangular matrix with diagonal

\[
(1, a_1, a_2, \ldots, a_{n-r-1}, \left(1 + \sum_{i=1}^{n-r} a_i\right)),
\]

when \(n < 2r\). Thus the Jacobian,

\[
J = \prod_{k=1}^{n-r-1} a_k \left(1 + \sum_{i=1}^{n-r} a_i\right),
\]

is nonzero for positive \(a\) when \(n < 2r\).

Observe that if \(n < 2r\) and \(b = 0\) then \(f_i = 0\) for all \(i\) if \(a\) is defined by

\[
a_1 = a_2 = \cdots = a_{r-1} = 1 \quad \text{and} \quad a_r = a_{r+1} = \cdots = a_n = \frac{1}{n - r + 1}.
\]

Since the Jacobian is nonzero for this choice of \(a\) (and \(b\)), by the Implicit Function Theorem there are differentiable (and hence continuous) functions \(a_1, a_2, \ldots, a_n\) of \(b_1, b_2, \ldots, b_n\) such that \(f_i = 0\) for all \(i\). Since \(a > 0\) and the functions \(a_i\) are continuous, for any \(b\) near \(0\), we can maintain \(f_i = 0\) for all \(i\) with some positive \(a\). Therefore, for \(n < 2r\), \(cA_{n,r}\) can have arbitrary characteristic polynomial and hence \(D_{n,r}\) is spectrally arbitrary.

If \(n = 2r\) the Jacobian matrix is the same as above except the matrix \(M\) has its \((r + 1, 1)\) entry equal to \(a_n\). In which case the Jacobian can be shown to be

\[
J = \prod_{k=1}^{n-r-1} a_k \left(1 + \sum_{i=1}^{n-r} a_i \cdot \frac{a_n}{a_1}\right).
\]
which is nonzero for positive \( a \), and when \( n = 2r \), if \( b = 0 \) and

\[
a_1 = \cdots = a_{r-1} = 1, \quad a_r = a_{r+1} = \cdots = a_{n-2} = \frac{-r + \sqrt{r^2 + 4}}{2} = a_n,
\]

and \( a_{n-1} = a_n^2 \), then \( a > 0 \) and \( f_i = 0 \) for all \( i \). Therefore it follows that \( D_{2r, r} \) is spectrally arbitrary and we have established the following theorem.

**Theorem 2.3.** If \( n \leq 2r \), then \( D_{n, r} \) is a spectrally arbitrary pattern.

Given Theorem 2.3, and given that Drew et al. [2] conjectured \( D_{n, 2} \) is a SAP for all \( n \geq 2 \), one might expect that \( D_{n, r} \) is a SAP for all \( n \geq r \). The corollary of the next result implies that this expectation does not hold.

A pattern \( S \) is potentially nilpotent if there is a matrix \( A \in Q(S) \) such that \( A^n = 0 \). (See for example [5,10].) Thus, if \( S \) is a SAP, then \( S \) is potentially nilpotent.

**Theorem 2.4.** If \( r \geq 3 \), then \( D_{2r+1, r} \) is not potentially nilpotent.

**Proof.** Suppose \( n = 2r + 1 \) and \( B \in Q(D_{n, r}) \). We may assume \( B \) has been scaled so that \( B_{n,n} = 1 \). We may also assume that all nonzero entries of \( B \) above the main diagonal are 1 (otherwise they can be adjusted to be 1 by suitable similarities). Thus, assume

\[
B = \begin{bmatrix}
-a_1 & 1 & 0 & \cdots & \cdots & 0 \\
-a_2 & 0 & 1 & \ddots & & \\
-a_3 & 0 & 0 & 1 & \ddots & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \\
-a_r & 0 & & & \ddots & \\
0 & -a_{r+1} & 0 & \cdots & 1 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
0 & \cdots & 0 & -a_n & 0 & \cdots & 0 & 1
\end{bmatrix}
\]

The characteristic polynomial of \( B \) is

\[
\det(xI - B) = (x - 1) \det(xI - A_{2r,r}^{(a-1)}) + a_n \det(xI - A_{r+1,r}^{(r+1)})
\]

and by Lemma 2.1 we have

\[
\det(xI - B) = (x - 1) \left[ \sum_{i=0}^{r-1} a_i x^{2r-i} + \sum_{i=r}^{2r} \sum_{k=i}^{r} a_{i-r} a_k x^{2r-i} \right]
\]

\[
+ a_n \left[ \sum_{i=0}^{r-1} a_i x^{r+1-i} + \sum_{i=r}^{r+1} \sum_{k=i}^{r+1} a_{i-r} a_k x^{r+1-i} \right]
\]
Therefore the coefficient of $x^r$ is nilpotent. By examining the coefficient of $x^{n-i}$, it follows that $a_i = a_{i-1}$ for $i = 1, \ldots, (r-1)$ and consequently

$$a_{r-1} = a_{r-2} = \cdots = a_1 = a_0 = 1.$$ 

Therefore the coefficient of $x^{r+1-i}$ reduces to $a_n a_i - a_{i-1} a_{r+i-1}$ for $i = 1, \ldots, (r-1)$. Thus

$$a_n = a_r = a_{r+1} = \cdots = a_{n-3}.$$ 

Since $r \geq 3$, setting the constant term equal to zero gives $a_{n-1} = a_n$, whereas the coefficient of $x$ gives

$$a_{n-2} = 3a_n^2 - a_n.$$  \hspace{1cm} (2)

On the other hand, the coefficient of $x^{n-r}$ forces

$$a_{n-2} = 1 - (n-r)a_n$$  \hspace{1cm} (3)

Solving (2) and (3) for $a_n > 0$ gives $a_n = (\sqrt{r^2 + 12} - r)/6$. But then by (3) we find that $a_{n-2} < 0$ when $r > 2$. This contradicts the fact that $B \in Q(\mathcal{D}_{2+1,r})$. Thus there is no $B \in Q(\mathcal{D}_{2+1,r})$ with $B$ nilpotent when $r \geq 3$. \hfill \Box
Corollary 2.5. \( D_{2r+1,r} \) is not a SAP when \( r \geq 3 \).

Using a similar argument, we can also show that \( D_{2r+2,r} \) is not a SAP when \( r \geq 3 \). In summary, for \( r \geq 3 \), we know that if \( \left\lfloor \frac{n+1}{2} \right\rfloor \leq r \leq n \) then \( D_{n,r} \) is a SAP, and if \( r = \left\lfloor \frac{n-1}{2} \right\rfloor \) then \( D_{n,r} \) is not a SAP. As noted earlier, \( D_{n,2} = T_n \) is known to be a SAP for \( n \leq 16 \). If \( r = 2 \) and \( n > 16 \), or if \( 2 < r \leq \left\lfloor \frac{n-3}{2} \right\rfloor \), then it is unknown whether or not \( D_{n,r} \) is a SAP.

3. Subpatterns and superpatterns

We say \( H \) is a subpattern of an \( n \times n \) pattern \( S \) if \( H = S \) or \( H \) is obtained from \( S \) by replacing one or more nonzero entries by a zero. If \( H \) is a subpattern of \( S \) then we also say \( S \) is a superpattern of \( H \). A sign pattern which is a SAP (or an IAP) is minimal, denoted MSAP (resp. MIAP), if no subpattern is a SAP (resp. IAP).

Before we discuss the subpatterns and superpatterns of \( D_{n,r} \), we need some further definitions.

A pattern \( S \) requires (resp. allows) a property \( P \) if every (resp. some) matrix \( A \in Q(S) \) has property \( P \). For example, we will use the fact that if a pattern \( S \) requires a positive eigenvalue, every matrix \( A \in Q(S) \) must have a positive eigenvalue and hence \( S \) would not be a SAP.

Theorem 3.1. Each of the sign patterns \( D_{n,r} \) with \( n \leq 2r \) is a minimal SAP.

Proof. Suppose \( S = [s_{ij}] \) is a subpattern of \( D_{n,r} \) and \( S \) is a SAP. We claim \( S = D_{n,r} \).

(1) \( s_{11} \neq 0 \) and \( s_{n,n} \neq 0 \), otherwise the trace of \( S \) is nonnegative or nonpositive.

(2) For \( 1 \leq i \leq n-1 \), \( s_{i,i+1} \neq 0 \), otherwise \( S \) is reducible with two of its irreducible submatrices requiring a nonzero eigenvalue.

(3) For \( 2 \leq i \leq r-1 \), \( s_{i,1} \) must be nonzero to allow the coefficient of \( x^{n-i} \) in the characteristic polynomial of any matrix realization of \( S \) to be arbitrary.

(4) \( s_{n-1,n-r} \neq 0 \) and \( s_{n,n-r+1} \neq 0 \), otherwise the determinant of \( S \) is not arbitrary.

(5) For \( 0 \leq i \leq n-r-2 \), \( s_{r+i+1} \neq 0 \), otherwise \( S \) is not potentially nilpotent. In particular, we argue if \( S \) is potentially nilpotent, then setting any \( s_{r+i+1} \) to zero forces \( s_{n,n-r+1} = 0 \), which is contrary to Case 4. Note that if \( A \in D_{n,r} \), then by scaling and similarity we can assume \( A = A_{n,r} \). If in addition \( A \) is nilpotent, then it follows that \( a_k = a_{k-1} \) for \( k = 1, \ldots, (r-1) \) by examining the coefficient of \( x^{n-k} \) in the characteristic polynomial of \( A \) given in Lemma 2.2. Thus

\[
a_{r-1} = a_{r-2} = \cdots = a_1 = a_0 = 1.
\]

Consequently (noting also that \( n \leq 2r \)) the coefficient of \( x^{n-k} \) reduces to \( a_{k-r}a_n - a_{k-r-1}a_{k-1} = a_n - a_{k-1} \) for \( k = r+1, \ldots, n-1 \). Thus, since \( A \) is
nilpotent, setting any $a_{k-1}$ to zero, $r + 1 \leq k \leq n - 1$, would force $a_n = 0$. That is, if $\mathcal{S}$ is potentially nilpotent and if $s_{r+i,i+1} = 0$ for any $i \in \{0, \ldots, n-r-2\}$, then $s_{n,n-r+1} = 0$.

Thus there are no proper subpatterns of $\mathcal{D}_{n,r}$ which are spectrally arbitrary. □

The technique from Drew et al. [2, Observation 10] that we used in the proof of Theorem 2.3 can be outlined as follows:

**Method 3.2** [2]. To show a pattern $\mathcal{S}$ is spectrally arbitrary: first find a matrix realization $A$ of the pattern $\mathcal{S}$ containing $n$ variables $a_1, a_2, \ldots, a_n$; then show that $A$ is nilpotent for some choice of $a = (a_1, a_2, \ldots, a_n) > 0$, call it $\hat{a}$; then show that the Jacobian of the coefficients of the characteristic polynomial of $A$ with respect to the variables $a_1, a_2, \ldots, a_n$ is nonzero for $a = \hat{a}$.

**Remark 3.3** [2, Observation 15]. If $\mathcal{S}$ is determined to be spectrally arbitrary via Method 3.2, then every superpattern of $\mathcal{S}$ is spectrally arbitrary.

**Corollary 3.4.** If $n \leq 2r$, every superpattern of $\mathcal{D}_{n,r}$ is spectrally arbitrary.

The $(2r-1)$-diagonal sign pattern $\mathcal{S}_{n,r}$ is an example of a superpattern of $\mathcal{D}_{n,r}$. It was shown in [6,9] that $\mathcal{S}_{n,n}$ and $\mathcal{S}_{n,n-1}$ respectively are inertially arbitrary. Britz et al. [1] demonstrate that $\mathcal{S}_{n,n}$ is not only an IAP but also a SAP. Corollary 3.4 demonstrates that $\mathcal{S}_{n,n-1}$ is also a SAP. In fact, we have the following corollary.

**Corollary 3.5.** If $n \leq 2r$, then $\mathcal{S}_{n,r}$ is a SAP (and hence also an IAP).

### 4. Generating more SAPs

The pattern $\mathcal{D}_{n,r}$ can be considered to be part of the larger class of patterns

\[
\mathcal{L}_{n,r} = \begin{bmatrix}
- & + & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
- & 0 & + & 0 & \cdots & \cdots & \cdots & \vdots \\
\vdots & 0 & 0 & + & \cdots & \cdots & \cdots & \vdots \\
- & \vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \vdots \\
\ast & 0 & \cdots & \vdots & \ddots & 0 & \cdots & 0 \\
\ast & \ast & 0 & \cdots & \ddots & + & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \cdots & \ddots & 0 + \\
\ast & \cdots & \ast & 0 & \cdots & 0 & \cdots & 0 & + 
\end{bmatrix}
\]
where \( * \in \{ +, -, 0 \} \). Within this larger class of patterns, we can demonstrate other matrices are spectrally arbitrary:

Let

\[
L_{n,r} = \begin{bmatrix}
-a_1 & 1 & 0 & 0 & \cdots & \cdots & 0 \\
-a_2 & 0 & 1 & 0 & & & \\
\vdots & 0 & 0 & 1 & & & \\
-a_{r-1} & \vdots & \ddots & \ddots & & & \\
-b_{r-1} & 0 & \ddots & \ddots & \ddots & \ddots & \\
-b_{r+1,1} & -b_{r+1,2} & 0 & \ddots & \ddots & 1 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
-b_{n,1} & \cdots & \cdots & -b_{n,n-r+1} & 0 & \cdots & 0 \\
\end{bmatrix},
\]

where \( a_i > 0 \) for \( 1 \leq i < r \). For the convenience of the next proof in the case when \( n = 2r \), we also let \( a_r = b_{r,1} \).

**Lemma 4.1.** If \( n \leq 2r \) and \( a_0 = 1 \), then the characteristic polynomial of \( L^{(n)}_{n,r} \) is

\[
\sum_{i=0}^{r-1} a_i x^{n-i} + \sum_{i=r}^{n} \sum_{w=0}^{n-i} a_w b_{j+i,j+w+1} x^{n-i}.
\]

**Proof.** Suppose \( r < n \). By expansion along the last row we note that if \( n \leq 2r \), then

\[
\det (x I - L^{(n)}_{n,r}) = \sum_{i=0}^{n-r+1} b_{n,i} \det (x I - C_{i-1}) + x \det (x I - L^{(n-1)}_{n-1,r}),
\]

since the cofactor of entry \( b_{n,i} \) is the determinant of a block lower triangular matrix with two diagonal blocks: \( C_{i-1} \) and a lower triangular matrix of order \( n - i \) with \((-1)^i\)'s on the diagonal. Hence by induction on \( n \) we find the characteristic polynomial of \( L^{(n)}_{n,r} \) is

\[
\sum_{i=1}^{n-r+1} \sum_{j=1}^{i-1} a_j b_{n,j} x^{i-1-j} + \sum_{i=0}^{r-1} a_i x^{n-i} + \sum_{i=r}^{n-1} \sum_{w=0}^{n-1-i} \sum_{j=0}^{i-1} a_w b_{j+i,j+w+1} x^{n-i}.
\]

Focusing on the first double sum we obtain

\[
\sum_{i=1}^{n-r+1} \sum_{j=0}^{i-1} a_j b_{n,j} x^{i-1-j} = \sum_{i=0}^{n-r} \sum_{j=0}^{i} a_j b_{n,i+1} x^{i-j}
\]
\[ \sum_{i=0}^{n-r} \sum_{j=0}^{n-r-i} a_j b_{n, j+i+1} x^i \]

\[ = \sum_{i=r}^{n} \sum_{w=0}^{n-i-1} a_w b_{n, n-i+w+1} x^{n-i} \]

from which the formula follows. □

Lemma 4.2. If \( n \leq 2r \), \( a_0 = 1 \) and \( a_{-1} = 0 \), then the characteristic polynomial of \( L_{n,r} \) is

\[ \sum_{i=0}^{r-1} (a_i - a_{i-1}) x^{n-i} + \left[ \sum_{j=0}^{n-r} b_{j+r, j+1} - a_{r-1} \right] x^{n-r} \]

\[ + \sum_{i=r+1}^{n} \sum_{j=0}^{n-i} \left[ \sum_{w=0}^{i-r-1} a_w b_{j+i, j+w+1} - \sum_{w=0}^{i-r-1} a_w b_{j+i-1, j+w+1} \right] x^{n-i}. \]

Proof. Using the linearity of the determinant, focusing on the last column, we note that \( \det(xI - L_{n,r}) = \det(xI - L_{n}^{(n)}) - \det(xI - L_{n-1}^{(n-1)}). \) The result follows from Lemma 4.1. □

We claim that \( L_{n,r} \) is nilpotent for \( n < 2r \) if and only if

\[ a_1 = a_2 = a_3 = \cdots = a_{r-1} = 1, \quad (5) \]

\[ \sum_{j=0}^{n-r} b_{j+r, j+1} = 1 \quad (= a_{r-1}), \quad (6) \]

and for \( (r+1) \leq i \leq n, \)

\[ \sum_{k=0}^{n-i} b_{k+i, k+1} + \sum_{k=n-i+2}^{n-r+1} b_{n, k} = \sum_{k=1}^{i-r} b_{l-1, k}. \quad (7) \]

To obtain (7), note that on using (5), the coefficient of \( x^{n-i} \), for \( (r+1) \leq i \leq n, \)

reduces to

\[ \sum_{j=0}^{i-r} \sum_{w=0}^{i-r-1} b_{j+i, j+w+1} - \sum_{w=0}^{i-r-1} b_{j+i-1, j+w+1} \]

when \( n < 2r. \) Upon expansion, this reduces the coefficient of \( x^{n-i} \) to

\[ \sum_{k=0}^{n-i} b_{k+i, k+1} + \sum_{k=2}^{i-r+1} b_{n, n-i+k} - \sum_{k=1}^{i-r} b_{l-1, k} \quad (8) \]
giving (7) when set to zero. It is helpful that the right side of Eq. (7) can be thought of as one less than the sum of the entries in row \((i - 1)\) of \(L_{n,r}\) and the left side can be visualized as the sum of the entries in the \(i\)-elbow:

\[
\begin{array}{cccc}
& b_{i,1} & & \\
& b_{i+1,2} & & \\
& \ddots & & \\
& b_{n,n-i+1} & \cdots & b_{n,n-r+1} \\
\end{array}
\]

We use this characterization of nilpotence to obtain spectrally arbitrary patterns in Theorem 4.4. We first provide a lemma, which will be used to argue we have a nonzero Jacobian. A sign pattern \(S\) is sign-nonsingular if every matrix \(A \in Q(S)\) is nonsingular.

**Lemma 4.3.** Suppose \(M\) is a sign pattern

\[
\begin{array}{ccccc}
* & \cdots & \cdots & * \\
- & * & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & * & * \\
0 & \cdots & 0 & - & +
\end{array}
\]

where \(* \in \{0, +\}\). If each row of \(M\) has at least one positive entry, then \(M\) is sign-nonsingular.

**Proof.** Suppose \(A = [a_{ij}]\) is of order \(n\) and \(A \in Q(M)\). Let \(A_k\) represent the \(k\)th column of \(A\). If \(n = 2\), then adding \(|a_{21}|A_2\) to \(A_1\) produces a matrix with the sign pattern

\[
\begin{array}{cc}
+ & * \\
0 & +
\end{array}
\]

Since this column operation would not change the determinant, and since \(\det(M') > 0\), it follows that \(\det(A) > 0\) and thus \(\det(M) > 0\). If \(n > 2\), then adding \(|[a_{n,n-1}]|A_n\) to column \(A_{n-1}\) produces a matrix with sign pattern

\[
\begin{array}{ccc}
N & & * \\
0 & \cdots & 0 & +
\end{array}
\]

where \(N\) is a pattern of order \(n - 1\) with the same properties as \(M\). By induction it follows that \(\det(N) > 0\). Since \(\det(M) = \det(M')\), we have \(\det(M) > 0\). \(\square\)

**Theorem 4.4.** Suppose \(n < 2r\), and \(B \in L_{n,r}\). If each row of \(B\) has exactly one negative and one positive entry, and each \(i\)-elbow, \((r + 1) \leq i \leq n\), contains at
least one nonzero (negative) entry then $\mathcal{B}$ is a SAP and every superpattern of $\mathcal{B}$ is a SAP.

**Proof.** Suppose that $\mathcal{B}$ satisfies the conditions in the theorem and that $A \in Q(\mathcal{B})$. For convenience, we will label the negative entry in row $i$ as $-a_i$ for $1 \leq i \leq n$ (as we did for $A_{n,r}$). For $(r+1) \leq i \leq (n-1)$, Eq. (7) determine that each $a_i$ is a positive linear combination of some subset of entries $a_j$ with $j > i$. Then the equations in (6) and (7) will have a positive solution $(\hat{a}_r, \hat{a}_{r+1}, \ldots, \hat{a}_n)$ and there is a vector $\hat{a} = (1, \ldots, 1, \hat{a}_r, \hat{a}_{r+1}, \ldots, \hat{a}_n)$ such that $A \in Q(\mathcal{B})$ is nilpotent.

We next find the Jacobian of the coefficients of the characteristic polynomial described in Lemma 4.2, as in Method 3.2. Focusing on the coefficients of $x^{n-r}$, $1 \leq i < r$, one sees that the Jacobian matrix is block triangular with upper left block $W$ of order $(r-1)$ and lower right block $M$ of order $(n-r+1)$. $W$ is a lower triangular matrix with ones on the main diagonal. Thus, the Jacobian is the determinant of $M$.

Since

\[
M = \begin{vmatrix}
\delta(f_r, f_{r+1}, \ldots, f_n) \\
\delta(a_r, a_{r+1}, \ldots, a_n)
\end{vmatrix},
\]

evaluating $M$ at $\hat{a}$ will be equivalent to first setting $a_1 = \cdots = a_{r-1} = 1$ before calculating the partials for $M$, and then evaluating the determinant of $M$ at $(\hat{a}_r, \hat{a}_{r+1}, \ldots, \hat{a}_n)$.

In this case,

\[
M = \begin{bmatrix}
1 & * & \cdots & \cdots & * \\
-1 & * & \ddots & & \\
0 & \ddots & \ddots & \ddots & \\
\vdots & \ddots & \ddots & * & * \\
0 & \cdots & 0 & -1 & 1
\end{bmatrix},
\]

where $* \in \{1, 0\}$. In particular, it follows from (8) that all subdiagonal entries of $M$ equal $-1$, since each row $i$ of $A$ contains exactly one negative entry, $-a_i$. The coefficient of $x^{n-r}$ determines that the first row of $M$ contains only nonnegative terms. Also, since each $i$-elbow of $A$, $(r+1) \leq i \leq n$, contains exactly one negative entry, it follows from (8) that $M$ is zero below the subdiagonal and has at least one positive entry in each row. By Lemma 4.3 it follows the sign pattern of $M$ is sign-nonsingular and hence the Jacobian is nonzero.

Therefore each pattern $\mathcal{B}$ is a SAP and by Remark 3.3, each superpattern of $\mathcal{B}$ is a SAP. □

**Examples 4.5.** The following patterns of order $n \geq 7$ are two of the many patterns which are SAPs by Theorem 4.4:
Example 4.6. Since we did not insist that $b_{i,j} > 0$ in the pattern $L_{n,r}$, we can use the characterization of nilpotency in Eqs. (5)-(7) to form other SAPs with some $b_{i,j} < 0$. The following is an example of such a pattern which can be shown to be spectrally arbitrary for all $n \geq 4$ using Method 3.2:

$$
\begin{pmatrix}
- + 0 & \cdots & 0 \\
- 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots \\
- 0 & \ddots & \ddots \\
- 0 & 0 & \ddots & + 0 \\
0 & 0 & \cdots & 0 & + \\
0 & 0 & \cdots & 0 & + \\
+ & - & 0 & \cdots & 0 & +
\end{pmatrix}
$$

5. Characterization of SAPs and IAPs of order 3

If $\mathcal{S}$ is an IAP of order 3, then it is not hard to see that $Q(\mathcal{S})$ allows a positive and negative principal minor of order $k$ for all $k = 1, 2, 3$. We refer to this as the minor conditions. The minor conditions, for example, indicate that an IAP cannot be sign-nonsingular.

We say a pattern $\mathcal{S}$ contains a negative 2-cycle if $s_{kjsjk} < 0$ for some $k \neq j$.

Lemma 5.1. If $\mathcal{S}$ is an IAP, then $\mathcal{S}$ contains a negative 2-cycle.

Proof. Suppose $\mathcal{S}$ is an IAP. Suppose $A \in Q(\mathcal{S})$ and $i(A) = (0, 0, n)$. Then the trace of $A$ is zero. If $E_2$ is the coefficient of $x^{n-2}$ in the characteristic polynomial of $A$ and $\pm b_1 i, \pm b_2 i, \ldots, \pm b_m i$ are the nonzero eigenvalues of $A$, then $E_2 = (\sum b_k^2)$. On the other hand, the sum of the 2-by-2 minors of $A$ is $E_2 = \sum_{k<j} a_{kk}a_{jj} - \sum_{k<j} a_{kj}a_{jk}$. Since $(1, 0, n-1) \in i(\mathcal{S})$, the pattern $\mathcal{S}$ (and hence $A$) must have at least one nonzero entry on its main diagonal and so $\sum a_{kk}^2 > 0$. Note that
\[ 2 \sum_{k<j} a_{kk}a_{jj} = (\text{tr}(A))^2 - \sum a_{kk}^2 < 0 \text{ if } \text{tr}(A) = 0. \] Since \( E_2 = (\sum b_k^2) \geq 0 \), we must have \( \sum_{k<j} a_{kj}a_{jk} < 0. \) □

Kirkland et al. [7, Lemma 5.1] showed that the pattern
\[ G = \begin{pmatrix} - & + & - \\ - & + & - \\ - & - & + \end{pmatrix} \]
requires a positive eigenvalue. By continuity, we have the following:

**Lemma 5.2.** If \( \mathcal{F} \) is a subpattern of \( \mathcal{G} \), then \( \mathcal{F} \) requires a nonnegative eigenvalue.

The following argument will demonstrate that if \( \mathcal{F} \) is an irreducible pattern of order 3 which contains a negative 2-cycle, and if \( \mathcal{F} \) satisfies the minor conditions, then either \( \mathcal{F} \) is equivalent to a subpattern of \( \mathcal{G} \) or \( \mathcal{F} \) is equivalent to a superpattern of one of the four matrices:

\[ D_{3,3} = \begin{pmatrix} - & + & 0 \\ - & 0 & + \\ - & 0 & + \end{pmatrix}, \quad \mathcal{U} = \begin{pmatrix} - & + & 0 \\ - & + & + \\ 0 & + & - \end{pmatrix}, \]
\[ D_{3,2} = \begin{pmatrix} - & + & 0 \\ - & 0 & + \\ 0 & - & + \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} - & 0 & + \\ - & 0 & + \\ - & + & + \end{pmatrix}. \]

Britz et al. [1] have also characterized the SAPs of order 3; they demonstrate that an irreducible pattern is a SAP if and only if it is potentially nilpotent and has at least one positive and one negative entry on its main diagonal.

**Theorem 5.3.** If \( \mathcal{F} \) is a pattern of order 3, then the following statements are equivalent:

1. \( \mathcal{F} \) is spectrally arbitrary.
2. \( \mathcal{F} \) is inertially arbitrary.
3. Up to equivalence, \( \mathcal{F} \) is a superpattern of \( D_{3,3}, D_{3,2}, \mathcal{U}, \) or \( \mathcal{V} \).

**Proof.** Drew et al. [2] showed that \( \mathcal{U} \) and \( D_{3,2} \) are a MSAP (and a MIAP) and each superpattern of these matrices are SAPs (and hence IAPs). In Section 3 we observed that each \( \mathcal{G}_{n,r} \), with \( n \leq 2r \), is a MSAP. That \( D_{3,3} \) is a MIAP follows from the minor conditions and the proof of Theorem 3.1 (1)–(4). Consider the matrix
\[ A = \begin{pmatrix} -a_1 & 0 & 1 \\ -a_2 & 0 & 1 \\ -a_3 & 1 & 1 \end{pmatrix} \in Q(\mathcal{F}). \]

Then \( \det(xI - A) = x^3 + (a_1 - 1)x^2 + (-a_1 + a_3 - 1)x - a_1 + a_2 \). If \( (a_1, a_2, a_3) = (1, 1, 2) \) then \( A \) is nilpotent and further, the Jacobian is nonzero for \( a = \ldots \)
Method 3.2. We observe that \( \mathcal{S} \) is a SAP (and hence an IAP) by Lemma 5.1. We have the following three cases to consider.

(a) Suppose \( s_{1,2} < 0 \). Using a signature similarity \((+, -, +)\) if necessary, we can assume

\[
\mathcal{S} = \begin{bmatrix}
- & + & * \\
- & * & * \\
* & * & +
\end{bmatrix},
\]

where \(* \in \{+, -, 0\}\). Since \( \mathcal{S} \) is irreducible, then up to a signature similarity \((+, +, -)\), we have \( s_{2,2} < 0 \) or \( s_{3,1} < 0 \).

(i) If \( s_{2,2} > 0 \) then \( \mathcal{S} \) is equivalent to a superpattern of \( \mathcal{D}_{3,2} \).

(ii) Suppose \( s_{2,3} > 0 \). If \( s_{3,1} > 0 \) then \( \mathcal{S} \) is equivalent to a superpattern of \( \mathcal{D}_{3,3} \) via transposition and signature similarity \((+, +, -)\). Assume \( s_{3,1} < 0 \).

(A) If \( s_{2,2} = 0 \) then \( \mathcal{S} \) is sign-nonsingular.

(B) If \( s_{2,2} < 0 \) and if \( s_{3,1} = 0 \) or \( s_{3,1} = 0 \) then \( \mathcal{S} \) is sign-nonsingular.

(C) If \( s_{2,2} < 0 \), \( s_{3,1} < 0 \) and \( s_{1,3} > 0 \) then \( \mathcal{S} \) is equivalent to a superpattern of \( \mathcal{V} \), via transposition (23) and signature similarity \((+, +, -)\).

(D) If \( s_{2,2} > 0 \), then \( \mathcal{S} \) is equivalent to a subpattern of \( \mathcal{G} \).

(iii) Suppose \( s_{2,3} = 0 \). Then since \( \mathcal{S} \) is irreducible, \( s_{3,1} > 0 \).

(A) If \( s_{3,1} < 0 \) and \( s_{2,2} < 0 \) then \( \mathcal{S} \) is equivalent to a superpattern of \( \mathcal{D}_{3,2} \) via permutation (12) and signature similarity \((-+, +, +)\).

(B) If \( s_{1,3} \leq 0 \) and \( s_{2,2} \geq 0 \) then \( \mathcal{S} \) is equivalent to a subpattern of \( \mathcal{G} \).

(C) If \( s_{3,1} = 0 \) and \( s_{2,2} < 0 \) then \( \mathcal{S} \) is sign-nonsingular.

(D) If \( s_{3,1} > 0 \) and \( s_{2,2} \leq 0 \) then \( \mathcal{S} \) is sign-nonsingular.

(E) If \( s_{3,1} > 0 \) and \( s_{2,2} > 0 \) then \( -\mathcal{S} \) is equivalent to a superpattern of \( \mathcal{G} \), via permutation (12) and signature similarity \((+, +, -)\).

(b) Suppose \( s_{3,1} < 0 \). Then we can assume \( s_{3,2} = 0 \), otherwise \( \mathcal{S} \) is described in Case a (after a signature similarity \((+, +, -)\) if \( s_{3,2} > 0 \)).

(i) Suppose \( s_{2,3} > 0 \). Then \( \mathcal{S} \) is a superpattern of \( \mathcal{D}_{3,3} \).

(ii) Suppose \( s_{2,3} < 0 \). Via transpose and signature similarity \((-+, +, +)\), we see \( \mathcal{S} \) is equivalent to the pattern in Case 1a(iii).
(iii) Suppose \( s_{2,3} = 0 \). Then \( s_{1,3} \neq 0 \) since \( \mathcal{S} \) is irreducible.
(A) If \( s_{1,3} > 0 \) and \( s_{2,2} = 0 \) then \( \mathcal{S} \) is sign-nonsingular.
(B) If \( s_{1,3} > 0 \) and \( s_{2,2} < 0 \) then \( \mathcal{S} \) is equivalent to \( \mathcal{S}_{3,2} \) via permuta-

(2) Suppose \( s_{2,3}s_{3,2} < 0 \). Then \( \mathcal{S} \) is equivalent to the pattern considered in Case 1 via permutation \((3,2)\).

(3) Suppose \( s_{1,3}s_{3,1} < 0 \). Then, up to signature similarity \((+, +, -)\),

\[ \mathcal{S} = \begin{bmatrix} - & * & + \\ * & * & * \\ - & * & + \end{bmatrix}, \]

where \(* \in \{+, -, 0\}\). If \( s_{2,2} > 0 \), then via permutation \((2,3)\) we get a pattern in Case 1. If \( s_{2,2} < 0 \), then via permutation \((1,2)\) we get a pattern in Case 2.

We first note that if \( A \in Q(\mathcal{V}) \) and \( P \) is a reverse permutation matrix, then
\( P(-A)P^T \in Q(\mathcal{V}) \). Hence, we have the following observation:

**Remark 5.4.** If \( (n_1, n_2, n_3) \in i(\mathcal{V}) \) then \( (n_2, n_1, n_3) \in i(\mathcal{V}) \).

The following are matrix examples \( A \in Q(\mathcal{V}) \) with inertia triples \((0, 0, 4), (1, 0, 3), (1, 1, 2), (2, 0, 2), (2, 2, 0), (2, 1, 1), (3, 0, 1), (3, 1, 0), \) and \((4, 0, 0)\) respectively:
Using these examples and Remark 5.4 we see that \( i(\mathcal{N}) \) contains each triple \((n_1, n_2, n_3)\) such that \( n_1 + n_2 + n_3 = 4 \). Thus \( \mathcal{N} \) is an IAP.

But we claim that \( \mathcal{N} \) is not a SAP: Let \( B = [b_{i,j}] \) and \( C = [c_{i,j}] \) be 2-by-2 matrices with positive entries and \( A \in Q(\mathcal{N}) \) with

\[
A = \begin{bmatrix}
    b_{1,1} & b_{1,2} & 0 & 0 \\
    0 & 0 & -c_{1,1} & -c_{1,2} \\
    b_{2,1} & b_{2,2} & 0 & 0 \\
    0 & 0 & -c_{2,1} & -c_{2,2}
\end{bmatrix}.
\]

Then if \( x^4 - E_1 x^3 + E_2 x^2 - E_3 x + E_4 \) is the characteristic polynomial of \( A \), we have \( E_3 = c_{1,1} \det(B) - b_{2,2} \det(C) \) while \( E_4 = -\det(B) \det(C) \). Thus \( E_4 \leq 0 \)
when \( E_3 = 0 \). Therefore \( \mathcal{N} \) is not a SAP since the characteristic polynomial of \( A \) cannot have arbitrary coefficients.

Our final example below, albeit reducible, will also demonstrate that not all IAPs are spectrally arbitrary. But we would first like to note that if \( \mathcal{I} \) is an IAP then it does not follow that each irreducible component of \( \mathcal{I} \) is an IAP. We justify this remark with the pattern \( \mathcal{F} \oplus \mathcal{F}_2 \), where

\[
\mathcal{F} = \begin{bmatrix}
    + & + & 0 & 0 \\
    0 & 0 & - & - \\
    + & + & 0 & 0 \\
    0 & 0 & - & - \\
    0 & 0 & 0 & +
\end{bmatrix}
\quad \text{and} \quad
\mathcal{F}_2 = \begin{bmatrix}
    - & + \\
    - & +
\end{bmatrix}.
\]

Using Matlab, we were able to find matrices \( A \in Q(\mathcal{F}) \) with inertia triples \((5, 0, 0), (0, 5, 0), (0, 0, 5), (3, 2, 0), (0, 3, 2), (2, 0, 3), (1, 2, 2), (2, 1, 2), \) and \((2, 2, 1)\). Using these triples, and the fact that \( \mathcal{F}_2 \) is inertially arbitrary, we can obtain a matrix \( B \in Q(\mathcal{F} \oplus \mathcal{F}_2) \) with \( i(B) = (a, b, c) \) for any specified triple \((a, b, c)\) with \( a + b + c = 7 \). Thus \( \mathcal{F} \oplus \mathcal{F}_2 \) is an IAP.

On the other hand \( \mathcal{F} \) is not an IAP: Suppose \( B \in Q(\mathcal{F}) \). Up to scaling and similarity we can assume

\[
\begin{bmatrix}
    1 & 1 & 0 & 0 \\
    0 & -1 & -1 \\
    1 & 1 & 0 & 0 \\
    0 & -1 & -1
\end{bmatrix},
\begin{bmatrix}
    2 & 2 & 0 & 0 \\
    0 & 0 & -1 & -1 \\
    1 & 1 & 0 & 0 \\
    0 & 0 & -1 & -1
\end{bmatrix},
\begin{bmatrix}
    1 & 2 & 0 & 0 \\
    0 & 0 & -1 & -1 \\
    1 & 1 & 0 & 0 \\
    0 & 0 & -1 & -1
\end{bmatrix},
\begin{bmatrix}
    2 & 2 & 0 & 0 \\
    0 & 0 & 0 & -1 \\
    1 & 1 & 0 & 0 \\
    0 & 0 & 0 & -1
\end{bmatrix},
\begin{bmatrix}
    2 & 2 & 0 & 0 \\
    0 & 0 & 0 & -1 \\
    1 & 1 & 0 & 0 \\
    0 & 0 & 0 & -1
\end{bmatrix},
\begin{bmatrix}
    3 & 3 & 0 & 0 \\
    0 & -2 & -1 \\
    3 & 4 & 0 & 0 \\
    0 & -2 & -1
\end{bmatrix}
\]
Where \( a, b, c, d \) and \( f \) are all positive. Then the characteristic polynomial of \( B \) is
\[
x^5 + (d - 1)x^4 + (b - d)x^3 + (a - bc + bd - b)x^2 \\
+ (ad - bd + bc - ac + bcf)x + cf(a - b).
\]
If the constant term is zero (that is, \( a = b \)) then the coefficient of \( x \) is positively signed. Thus \((1, 1, 3) \notin \mathcal{I}(\mathbb{F}) \) and \( \mathbb{F} \) is not an IAP.

The analysis of the characteristic polynomial also indicates that \( B \) cannot be nilpotent and consequently \( \mathbb{F} \oplus \mathbb{F}_2 \) is not potentially nilpotent. Therefore, while every SAP is potentially nilpotent, the weaker condition of being an IAP is insufficient to be potentially nilpotent.

Acknowledgements

The authors thank the Natural Sciences and Engineering Research Council of Canada for undergraduate research grant USRA270973 and discovery grant 203336, respectively.

References