CLIQUE PARTITIONS OF THE COCKTAIL PARTY GRAPH

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Let $T_v$ denote the complement of a perfect matching in the complete graph on $v$ vertices, $v$ even, and let $cp(T_v)$ be the minimum number of cliques needed to partition the edge-set of $T_v$. We prove that $cp(T_v) > v$ for $v \geq 8$ and give a design characterization of the cases where equality holds. We also show that, asymptotically, $cp(T_v) \leq v \log \log v$.

1. Terminology

For our purposes, graphs have no loops or multiple edges. Aside from this, our terminology follows that of Bondy and Murty [1]. A clique is a complete subgraph. A clique partition of a graph $G$ is a family $\mathcal{C}$ of cliques of $G$ such that each edge of $G$ is in precisely one member of $\mathcal{C}$. The clique partition number, $cp(G)$, is the minimum of the cardinalities of the clique partitions of $G$.

Let $T_v$ denote the graph on $v=2n$ vertices $u_i$, $i=1, 2, \ldots, v$ with $u_i$ nonadjacent to $u_{i+n}$ for each $i = 1, 2, \ldots, n$. All other pairs of vertices are adjacent. This unique $(v-2)$-regular graph on $v$ vertices is called the cocktail party graph.

2. A lower bound on the clique partition number

We first show that, for $v \geq 8$, a clique partition of $T_v$ must have cardinality at least $v$. Our technique of assigning indeterminates to vertices and constraints to cliques was motivated by Tverberg's proof [13] of the Graham-Pollak theorem [4, 5]:

At least $v - 1$ complete bipartite subgraphs are needed to partition the edge set of $K_v$, the complete graph on $v$ vertices.

See also [9].
Theorem 1. If \( v = 2n \geq 8 \), then \( cp(T_v) \geq v \).

Proof. Let \( C = \{ C_1, C_2, \ldots, C_b \} \) be a clique partition of \( T_v \). We wish to show that \( b \geq v \). For each \( i = 1, 2, \ldots, v \) let \( r_i \) be the number of cliques in \( C \) which contain the vertex \( u_i \). Since non-adjacent vertices are in different cliques, \( r_i \geq 2 \) for each \( i \). We may actually assume that \( r_i \geq 3 \) for each \( i \). For if some vertex \( u \) were in precisely two cliques \( C, C' \) of \( C \), then since no two of the \( (n - 1)(n - 2) \) edges in \( T_v \) from \( C - u \) to \( C' - u \) could be in the same clique of \( C \), we would have \( b \geq (n - 1)(n - 2) + 2 \geq 2n = v \), as required.

Now, for each \( i = 1, 2, \ldots, v \) associate the indeterminate \( x_i \) to the vertex \( u_i \), and for each \( j = 1, 2, \ldots, b \) let \( L_j \) be the sum of the indeterminates associated with the vertices in \( C_j \). Since each edge of \( T_v \) is in precisely one clique of \( C \),

\[
\sum_{j=1}^{b} L_j^2 = \left( \sum_{i=1}^{v} x_i \right)^2 + \sum_{i=1}^{v} (r_i - 1)x_i^2 - 2 \sum_{i=1}^{v} x_ix_{i+1}.
\]

Since \( 2x_ix_j \leq x_i^2 + x_j^2 \), this implies that

\[
\sum_{j=1}^{b} L_j^2 \geq \left( \sum_{i=1}^{v} x_i \right)^2 + \sum_{i=1}^{v} (r_i - 2)x_i^2.
\]

Consider the system of \( b \) homogeneous linear equations \( L_j = 0, j = 1, 2, \ldots, b \) in the \( v \) indeterminates \( x_i, i = 1, 2, \ldots, v \). Since we may assume that \( r_i \geq 3 \) for each \( i \), it follows from the inequality above that the only solution of the system is the trivial one. Therefore, \( b \geq v \). \[ \square \]

Remarks. Let \( M \) be the \( v \times b \) vertex-clique incidence matrix associated with \( C \): \( M_{ij} = 1 \) if \( u_i \in C_j \), and \( M_{ij} = 0 \) otherwise. The second half of the proof of Theorem 1 also follows by applying a result of Ryser [11, Theorem 2.1] to the subsets of a \( b \)-set that are associated with the \( v \) rows of \( M \). Conversely, the indeterminate technique of Theorem 1 can be extended to clique partitions of arbitrary graphs to yield new proofs [3] of Theorems 2.1 and 2.2 in [11].

3. The case of equality: a connection with designs

Associating clique partitions with matrices as in the remarks above, it follows from [11, Theorem 1.1] that \( cp(T_v) = v \) if and only if there is a \( v \times v \) \((0 - 1)\)-matrix \( M \) such that:

1. Each row and column of \( M \) has the same number of 1's;
2. \( M \) has no \( 2 \times 2 \) submatrix whose entries are all 1; and
3. Each row of \( M \) is orthogonal to precisely one other.

Such a matrix is an example of a symmetrical group divisible design [2]. If \( k \) is the number of 1's in each row and column of \( M \), then \( v = k(k - 1) + 2 \). Results of
Bose and Connor [2, p. 381] give additional constraints on \( M \):

(a) If \( n \) is even, \( k - 2 \) must be a perfect square and, if further, \( n = 2 \mod 4 \), then \( (k, -1)_p = 1 \) for all odd primes \( p \). (Here, \((,)_p\) denotes the Hilbert norm residue symbol [2, p. 375]).

(b) If \( n \) is odd, then \( k \) is a perfect square and if \( \alpha = \binom{n}{2} \), then \((-1)^{\alpha 2}, k - 2)_p = 1 \) for all odd primes \( p \).

The first seven values of \( v \) satisfying (a) and (b) are \( v = 2, 4, 8, 14, 32, 74, 112 \). Examples in [10, p. 47] give equality for \( v = 4, 8, 14 \). We'll see below that equality fails for \( v = 32 \). It is not known if equality holds for any larger value of \( v \).

If \( cp(T_v) = v \), then by conditions 1, 2, 3 above, there is a partition of the edge-set of \( T_v \) into cliques of size \( k \) such that each vertex is in \( k \) of the cliques. Therefore, each clique is disjoint from precisely \( v - k(k - 1) = 1 \) other clique in the partition. Each vertex in the first clique must be non-adjacent to a vertex in the disjoint clique; otherwise, there would be more than \( k(k - 1) \) edges between the cliques, and so more than \( v = k(k - 1) + 2 \) cliques in the partition. Consequently, if the rows and columns of the matrix \( M \) above are grouped into orthogonal pairs, we obtain an \( n \times n \) array whose entries are the \( 2 \times 2 \) matrices:

\[
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

Mullin and Stanton [7, Theorem 1] have observed that replacing these matrices by 0, 1, and \(-1\), respectively, yields an \( n \times n \) matrix \( M' \) with the following properties:

1'. Each row of \( M' \) has \( k \) non-zero entries;

2'. each row of \( M' \) is orthogonal to every other row; and,

3'. if the \(-1\) entries of \( M' \) are replaced by \(+1\), then pairs of distinct rows have a common inner product \( \lambda \).

An \( n \times n \) matrix \( M' \) with entries 0, 1, and \(-1\) which satisfies 1' and 2' is called a weighing matrix. If \( M' \) also satisfies 3', it is called a balanced weighing matrix with parameters \((n, k, \lambda)\).

For the matrix \( M' \) obtained from \( M \) above, \( \lambda = 2 \). Conversely, a balanced weighing matrix \( M' \) with parameters \((n, k, 2)\) yields a \( v \times v \) matrix \( M \) satisfying 1, 2, and 3. This gives the following theorem.

**Theorem 2.** If \( v = 2n \), then \( cp(T_v) = v \) if and only if there is an \( n \times n \) balanced weighing matrix with \( \lambda = 2 \).

Schellenberg [12] has shown that there is no \( 16 \times 16 \) balanced weighing matrix with \( \lambda = 2 \). Therefore \( cp(T_{32}) > 32 \). The next case where equality may hold is \( v = 74, k = 9 \).

**Problem.** Does \( cp(T_{74}) = 74 \)? Equivalently, is there a balanced weighing matrix with parameters \((37, 9, 2)\)?
4. Upper bounds

For our estimates, we define $T_v$ for odd $v \geq 1$ to be the graph obtained from $T_{v+1}$ by deleting one vertex. Then, $T_v$ is an induced subgraph of $T_{v+1}$ for all $v$, and so $cp(T_v)$ is monotone in $v$. An argument similar to that in Theorem 1 shows that $cp(T_v) \geq v$ for all odd $v \geq 7$. Theorem 1.1 of Ryser [11, p. 60] can then be used to show that $cp(T_v) > v$ for all odd $v \geq 9$. Consequently, for $v \geq 9$, there are no new cases for which equality holds.

Using the results of the previous section and the remarks above, we were able to obtain, with some effort, exact values of $cp(T_v)$ for $v \leq 14$, see Table 1.

<table>
<thead>
<tr>
<th>$v$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>$cp(T_v)$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>7</td>
<td>8</td>
<td>10</td>
<td>11</td>
<td>14</td>
<td>14</td>
<td>14</td>
<td>14</td>
</tr>
</tbody>
</table>

Orlin [8, p. 411] has conjectured that $\lim_{n \to \infty} cp(T_{2n})/n = 1$. Theorem 1 suggests the following revision.

**Conjecture.** $\lim_{v \to \infty} cp(T_v)/v = 1$.

We do not even know if the limit exists. We conclude with some upper bounds. The logarithms are base 2.

**Theorem 3.** Let $v = p^{2^k}$, where $p$ is a prime and $k$ is a positive integer. If $p = 2$, then $cp(T_v)/v \leq \log \log v$. If $p$ is odd, then $cp(T_v)/(v - 1) \leq \log \log v + cp(T_v)/(p - 1) - \log \log p$.

**Proof.** Let $P$ be an affine plane of order $q$, where $q$ is a power of $p$. Then $P$ has $q^2$ points, $q^2 + q$ lines, $q$ points on each line, and $q$ lines in each parallel class. Associate the points of $P$ with the vertices of the complete graph $K_{q^2}$, and associate each line in $P$ with a $q$-clique in $K_{q^2}$. Since each pair of points is on one line, the cliques partition the edge-set of $K_{q^2}$.

Suppose that $p$ is odd. Form a graph isomorphic to $T_{q^2}$ by replacing each of the $q$-cliques associated with the lines of some parallel class by a copy of $T_q$; do this so that the $q$ distinguished vertices in the graphs $T_q$ are associated with a line $L$ in $P$. Replace the $q$-clique associated with $L$ by another $T_q$. The remaining $q^2 - 1$ of the $q$-cliques, together with minimum clique partitions of each of the $q + 1$ graphs $T_q$, give a clique partition of $T_{q^2}$. Thus, $cp(T_{q^2}) \leq (q^2 - 1) + (q + 1)cp(T_q)$, or

$$cp(T_{q^2})/(q^2 - 1) \leq 1 + cp(T_q)/(q - 1),$$

whenever $q$ is a power of the odd prime $p$. In particular, if $q = p^{2^k}$ and $h(k) = cp(T_q)/(q - 1)$, then $h(k + 1) \leq 1 + h(k)$ for all $k$. Thus $h(k) \leq k + h(0)$. This gives the result for $p$ odd.
When \( q \) is a power of \( p = 2 \), a simpler argument shows that
\[
\frac{cp(T_q)}{q^2} \leq 1 + \frac{cp(T_q)}{q}.
\tag{2}
\]
Since \( cp(T_2) = 0 \), this implies the result for \( p = 2 \). \( \square \)

Note that Theorem 3 gives an upper bound for only some \( v \). The following argument, due to a referee, gives an upper bound for all \( v \).

For \( q = 2^k \), let \( f(k) = \frac{cp(T_q)}{q} \). By inequality (2),
\[
f(2k) \leq 1 + f(k).
\tag{3}
\]
Constructions similar to those in Theorem 3 give other inequalities. Delete \( q \) of the \( 2q \) lines in one parallel class of an affine plane of order \( 2q \). In a second parallel class, replace each of the \( 2q \) lines (now on \( q \) points) by graphs \( T_q \). This gives
\[
f(2k + 1) < 2 + f(k).
\tag{4}
\]
Deleting \( \frac{1}{2}q \) of the \( q \) lines in a parallel class of an affine plane of order \( q \) and replacing each of the remaining \( \frac{1}{2}q \) lines by graphs \( T_q \) gives
\[
f(2k - 1) \leq 2 + f(k).
\tag{5}
\]
Advancing through the powers of two with these inequalities, each application of (4) or (5) contributes 2 and each application of (3) contributes 1. Since odd powers can be preceded by even powers by one of (4) or (5), a contribution of 2 can be preceded by one of 1. Thus, each of the roughly \( \log \log q \) steps needed to reach \( q \) from \( 2^1 \) contributes about 1.5 on average. A more detailed analysis shows that, indeed, \( cp(T_q) \leq 1.5q \log \log q \) whenever \( q \) is a power of 2. For arbitrary \( v \), choose \( q \) a power of 2 so that \( q \leq v < 2q \). Then \( cp(T_v) \leq cp(T_{2q}) \leq 1.5(2q) \log \log 2q \). Thus, for all \( v \),
\[
\frac{cp(T_v)}{v} \leq 3v \log \log 2v.
\tag{6}
\]
Using an estimate on the difference between consecutive primes, we can improve (6) for large \( v \).

**Theorem 4.** For each \( \varepsilon > 0 \), \( cp(T_v) < (1 + \varepsilon) v \log \log v \) for all \( v \) sufficiently large.

**Proof.** Given \( v \), let \( d \) be the largest even integer such that \( d \leq \sqrt{v} \) and let \( e \) be the smallest integer such that \( de \geq v \). Then \( e \leq d + 5 \). Let \( p \) be the smallest prime such that \( p \geq e \). In an affine plane of order \( p \), identify lines with cliques as in Theorem 3. Delete all but \( d \) of the \( p \) lines in one parallel class. In a second parallel class, replace \( e \) of the \( p \) lines (now with \( d \) points each) by graphs \( T_d \) and delete the points on the remaining \( p - e \) lines of the class. The \( (p^2 + p) - (p - d) - p \) lines/cliques remaining, together with minimum clique partitions of the \( e \)
graphs $T_d$, yield a clique partition of $T_{ed}$. As $v \leq ed$,

$$cp(T_v) \leq cp(T_{ed}) \leq (p^2 - p + d) + ecp(T_d) < p^2 + (d + 5)cp(T_d).$$

For positive real $x$, let $c(x) = cp(T_v)$, where $v$ is the integer part of $x$. Then $c(x)$ is monotone increasing and so

$$c(x) < p^2 + (\sqrt{x} + 5)c(\sqrt{x}).$$  \hfill (7)

Let $q$ be the largest prime less than $p$. Then $q \leq e - 1 \leq \sqrt{x} + 4$. By an estimate on consecutive prime differences (see [6]), $p < (1 + \varepsilon_p)(q + q^{0.6})$ where $\varepsilon_p \to 0$ as $p \to \infty$. Let $\varepsilon > 0$ be given. Since $c(\sqrt{x})/x \to 0$ as $x \to \infty$, substituting these estimates in (7) gives

$$c(x) < (1 + \frac{1}{2}\varepsilon)x + \sqrt{x} c(\sqrt{x})$$  \hfill (8)

for $x$ sufficiently large. Let $h(z) = c(x)/x$, where $x = 2^z$. By (8), $h(z) < 1 + \frac{1}{2}\varepsilon + h(z - 1)$ for all $z$ sufficiently large. Thus there is a constant $M$ such that $h(z) < (1 + \frac{1}{2}\varepsilon)z + M$ for all $z$. This implies the result stated. \hfill \square

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