Some Basic Properties of Vector Sequence Spaces

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Introduction

A vector sequence space (v.s.s.) $S(E)$ is a linear space of sequences of vectors chosen from a vector space $E$ over the scalar field of reals or complexes. This paper will be devoted to a study of some of the basic properties of a v.s.s. $S(E)$ with a locally convex Hausdorff topology $T$. In a subsequent paper (Gregory [3]), properties of $(S(E), T)$ such as barrelledness, reflexivity, and Montelness will be investigated. $(S(E), T)$ has been analyzed in the case where $E$ is the scalars by Zeller [10]. The present investigation was mostly motivated by a study of the “vollkommene Räume” of Köthe, which were first considered from the point of view of locally convex vector spaces in Köthe [5]. In the analogous case for v.s.s., we assume that a dual pair $\langle E, E' \rangle$ of vector spaces is given and examine a dual pair $\langle S(E), S'(E') \rangle$ of v.s.s. with a bilinear form induced by summation. Consequently, if a topology $T$ on $S(E)$ arises out of such a duality, we say that $(S(E), T)$ is of Köthe type. For example, the “verallgemeinerte vollkommene Folgenräume” of Pietsch [8], and the spaces $F(E)$ constructed by Cac [1] are of Köthe type.

An examination of v.s.s. from the viewpoint of dual pairs has also been made by Gribanov [4]. In that paper, $S(E)$ and $S'(E')$ are assumed perfect, and most of the proofs require an assumption of sequential completeness on $E$ or $E'$. The few results that the papers have in common are given here in a slightly more general setting.

In Section 1, the commonly accepted definitions used in scalar sequence spaces are extended to v.s.s. In Section 2, properties of projections are considered in a manner general enough to include the natural projections on $S(E)$, as well as a mapping used in Gregory [3]. In Sections 3 and 5, well-known properties of scalar sequence spaces are assumed, and analogous properties are obtained for the general case. In Section 4, the characterizations of compact subsets considered by Pietsch ([8], p. 334) are extended to $(S(E), T)$. Section 6, on completeness of v.s.s., emphasizes the value of the solid topologies. In Section 8, it is shown that if a matrix of weakly continuous linear maps defines a transformation, then the transformation is often weakly continuous. This is a generalization of a result due originally to Köthe and Toeplitz ([7], p. 206), and is finer than the result obtained by Gribanov ([4], Theorem 12, p. 66). In Section 9, we form a v.s.s. $\lambda(E, T)$ out of a scalar sequence space $\lambda$ and a locally convex space $(E, T)$. If $\lambda$ is perfect, and $T$ is the weak topology, then $\lambda(E, T)$ is the “verallgemeinerte vollkommene Folgenraum” of Pietsch, and is perfect in the sense to be defined here. If $E$ is normed, then $\lambda(E, T)$ is the space $F(E)$ constructed in Cac [1].
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1. Definition and Notation

Let $(E, E')$ be a dual pair of vector spaces over the field $K$ of real or complex scalars. We assume that $E$ and $E'$ distinguish one another by the bilinear form $\langle \cdot, \cdot \rangle$. Let $\mathcal{A}$ be a class of weakly bounded subsets of $E'$ with union equal to $E'$. By the polar topology $T(E', E)$ on $E$ of $\mathcal{A}$-convergence, we mean that topology on $E$ having as neighborhood subbasis at the origin the class of all sets of the form $A^\circ = \{x \in E : |\langle x, y \rangle| \leq 1 \text{ for all } y \in A\}$, where each $A$ is in $\mathcal{A}$. $T_s(E', E), T_k(E', E)$ and $T_b(E', E)$ will denote, respectively, the weak, Mackey, and strong topologies on $E$ induced by $E'$. The topological dual of $(E, T)$ will be denoted by $(E, T)'$.

By a vector sequence space (v.s.s.) $S(E)$ over $E$, we mean a set $S(E)$ of sequences $(x_n) = (x_1, x_2, \ldots)$ of vectors $x_n$ from $E$ that forms a linear space over $K$ under the usual componentwise operations. We also assume that $S(E)$ contains $\phi(E)$, the space of all sequences of vectors from $E$ with only a finite number of entries not equal to the zero vector. A v.s.s. $S'(E')$ over $E'$ is defined similarly. The zero vectors in $E, E', S(E)$, and $S'(E')$ will be denoted by 0, 0', (0), and (0'), respectively.

Let $\omega(E)$ denote the v.s.s. of all sequences of vectors from $E$. Then $\phi(E) \subseteq S(E) \subseteq \omega(E)$ always holds. Similarly, $\phi(E') \subset S'(E') \subset \omega(E')$ always holds.

A subset $X$ of $\omega(E)$ is called solid if $(x_n) \in X$ implies that $(c_n x_n) \in X$ for all scalar sequences $(c_n)$ such that $|c_n| \leq 1$ for all $n$. $X = \{(c_n x_n) : (x_n) \in X, c_n \in K, |c_n| \leq 1\}$ is the smallest solid set containing $X$ and is called the solid hull of $X$. A subset $X$ of $S(E)$ is called solid in $S(E)$ if $X \cap S(E) = X$. $X \cap S(E)$ is called the solid hull of $X$ in $S(E)$. The solid hull of a singleton $\{(c_n x_n)\}$ is denoted by $|(x_n)|$. Solid and solid hull are defined similarly in $\omega(E')$.

The $\beta$-dual $X^\beta$ of a subset $X$ of $\omega(E)$ is defined by

$$X^\beta = \{(y_n) \in \omega(E') : |\sum \langle x_n, y_n \rangle| < \infty \text{ for all } (x_n) \in X\}.$$ 

$Y^\beta$ is defined similarly for a subset $Y$ of $\omega(E')$. Note that $\omega(E)^\beta = \phi(E')$ and $\phi(E)^\beta = \omega(E')$. Note also that $\beta$ is an inclusion reversing map and thus $X^\beta = X^{\beta\beta\beta}$ for all $X$ contained in $\omega(E)$. Also, $X^\beta$ is a v.s.s. over $E'$.

The $\alpha$-dual $X^\alpha$ of a subset $X$ of $\omega(E)$ is defined by

$$X^\alpha = \{(y_n) \in \omega(E') : \sum |\langle x_n, y_n \rangle| < \infty \text{ for all } (x_n) \in X\}.$$ 

$Y^\alpha$ is defined similarly for a subset $Y$ of $\omega(E')$. The map $\alpha$ is also inclusion reversing and $X^\alpha = X^{\alpha\alpha\alpha}$ for all $X$ contained in $\omega(E)$. Moreover, $X^\alpha$ is a solid v.s.s. over $E'$. Note that if $X$ is solid, then $X^\alpha = X^\beta$.

A subset $X$ of $\omega(E)$ (resp. $Y$ of $\omega(E')$) will be called perfect if $X = X^{\alpha\alpha}$ (resp. $Y = Y^{\alpha\alpha}$). From the previous paragraph, we see that if $X$ is any subset of $\omega(E)$ then $X^\alpha$ and $X^{\alpha\alpha}$ are always perfect and $X^{\alpha\alpha\alpha}$ is the smallest perfect subset of $\omega(E)$ containing $X$.

$(S(E), T)$ will denote a locally convex topological v.s.s. (l.c.t.v.s.s.); in other words, a v.s.s. $S(E)$ with a locally convex Hausdorff topology $T$ compatible with the linear structure. $T$ is called solid if $(S(E), T)$ has a base of neighborhoods of (0) which are solid in $S(E)$. 


If \( S(E), S'(E') \) are v.s.s. over \( E, E' \) respectively, and if \( S'(E') \subset S(E)^* \), then \( \langle S(E), S'(E') \rangle \) is a dual pair of vector spaces over the field \( K \) with bilinear form \( \langle (x_n), (y_n) \rangle = \sum \langle x_n, y_n \rangle \). \( S(E) \) and \( S'(E') \) distinguish one another since \( S(E) \supseteq \varphi(E) \) and \( S'(E') \supseteq \varphi(E') \). We shall assume throughout that \( \langle S(E), S'(E') \rangle \) denotes a dual pair of v.s.s. with \( S'(E') \subset S(E)^* \) and with the bilinear form just defined. We usually require either \( S(E) \) or \( S'(E') \) to be solid. In this case, the bilinear form is an absolutely convergent series for all \( (x_n) \) in \( S(E) \), \( (y_n) \) in \( S'(E') \); equivalently, \( S'(E') \subset S(E)^* \).

An l.c.t.v.s.s. \( (S(E), T) \) is said to be of Köthe type if there is a v.s.s. \( S'(E') \) such that \( T \) is a polar topology on \( S(E) \) of \( \mathscr{A} \)-convergence for the dual pair \( \langle S(E), S'(E') \rangle \). Such a topology \( T \) would then be determined by the seminorms:

\[
(x_n) \rightarrow \sup_{(y_n) \in \mathscr{A}} \left| \sum \langle x_n, y_n \rangle \right|, \quad A \in \mathscr{A}.
\]

If \( (S(E), T) \) is of Köthe type, then \( T \) is solid if and only if the members of the class \( \mathscr{A} \) determining \( T \) can be chosen to be solid in \( S'(E') \). Because \( S'(E') \) contains \( \varphi(E') \), it follows that such a topology \( T \) is determined by the seminorms:

\[
(x_n) \rightarrow \sup_{(y_n) \in \mathscr{A}} \sum \left| \langle x_n, y_n \rangle \right|, \quad A \in \mathscr{A}.
\]

In particular, given the dual pair of v.s.s. \( \langle S(E), S'(E') \rangle \), if the solid hulls in \( S'(E') \) of elements of \( S'(E') \) are weakly bounded (equivalently, if \( S'(E') \subset S(E)^* \)), then they determine a solid polar topology \( T_0(S'(E'), S(E)) \) on \( S(E) \) called the normal topology.

2. Projection Maps

The maps regarded as projections here will not in general map a vector space into itself, but for convenience, will be assumed to map onto another vector space. We will be interested mainly in the maps:

\[
P_i: S(E) \to E \text{ defined by } P_i[(x_n)] = x_i\]

and \( I_i: E \to S(E) \) defined by \( I_i(x) = (\theta, \theta, \ldots, \theta, x, \theta, \ldots) \).

However, we shall first consider the following more general situation.

Let \( L \) and \( E \) be vector spaces and let \( P: L \to E \) and \( I: E \to L \) be linear maps such that \( PI \) is the identity map on \( E \). Then \( I \) is injective, \( P \) is onto, and \( IP \) is a projection of \( L \) onto \( I(E) \) in the usual sense. If \( L \) is a Hausdorff topological vector space with topology \( T \), we give \( E \) the relative topology \( T_I = \{I^{-1}(U): U \in T\} \).

Note that \( I \) is \( T_I \), \( T \) continuous and \( P \) is \( T_I \), \( T \) open ([9], p. 19).

Also, if \( P \) is \( T_I \), \( T \) continuous then \( I \) is \( T_I \), \( T \) closed. For if \( P \) is continuous, \( I(E) \) has a topological supplement. Thus \( I(E) \) is closed since it is the kernel of a continuous projection. Consequently, \( I \) is a closed map.

In the particular case when \( (L, T) = (S(E), T) \) is an l.c.t.v.s.s., and \( I = I_I, P = P_I \), then \( P_I I_I \) is the identity map on \( E \). We let \( T_{II} = T_I \) be the relative topology on \( E \).

Proposition 2.1 Let \( (S(E), T) \) be an l.c.t.v.s.s. Then:

1. \( I_I \) is \( T_I \), \( T \) continuous and \( P_I \) is \( T_I \), \( T_I \) open
2. If \( P_I \) is \( T_I \), \( T_I \) continuous then \( I_I \) is \( T_I \), \( T_I \) closed
3. If \( T \) is solid, then \( P_I \) is \( T_I \), \( T_I \) continuous
Proof. (1) and (2) are particular cases of the results just discussed. (3) holds since $T_j = \{I_j^{-1}(U) : U \in T\}$, and $P_j^{-1}[I_j^{-1}(U)] \supset U$ if $U$ is solid.

Suppose now that $L$ and $E$ are members of the dual pairs $\langle L, L' \rangle$ and $\langle E, E' \rangle$ respectively, and that $P$ and $I$ are weakly continuous with adjoints $P^*$ and $I^*$. If we let $I' = P^*$ and $P' = I^*$, then $P'I' = (P'I)^*$ is the identity map on $E'$. Thus $P'$ and $I'$ act on $L'$ and $E'$ as $P$ and $I$ act on $L$ and $E$.

**Lemma 2.2** Let $P$ and $I$ be weakly continuous and let $T$ be the polar topology on $L$ of $\mathcal{A}$-convergence. Then:

1. $T_I$ is the polar topology on $E$ of $P'(\mathcal{A})$-convergence, where 
   $$P'(\mathcal{A}) = \{P'(A) : A \in \mathcal{A}\}.$$ 

2. $P$ is $T$, $T_I$ continuous if and only if the class $\mathcal{A}$ determining $T$ can be chosen so that $I'P'(A) \in \mathcal{A}$ whenever $A \in \mathcal{A}$.

**Proof.** Since $P'$ is weakly continuous and onto, $P'(\mathcal{A})$ is a class of weakly bounded sets covering $E'$. Also, for each $A$ in $\mathcal{A}$, $I^{-1}(A^o) = [I^*(A)]^o = [P'(A)]^o$. The result (1) follows from this equation if we recall that the $I^{-1}(A^o)$ generate $T_I$ and 
   $$P'(\mathcal{A}) = \{P'(A) : A \in \mathcal{A}\}.$$ 

For each $A$ in $\mathcal{A}$ we have $P^{-1}[P'(A)^o] = [P^*P'(A)]^o = [I'P'(A)]^o$. Suppose that $T$ is continuous. Then $[I'P'(A)]^o$ is in $T$ since $P'(A)^o$ is in $T_I$ by (1). Thus, $I'P'(A)$ can be assumed to be in $\mathcal{A}$. Conversely, if $I'P'(A)$ is in $\mathcal{A}$, then $P^{-1}[P'(A)^o]$ is open and $P$ is continuous since $T_I$ is the topology of $P'(\mathcal{A})$-convergence.

**Remark.** If $\mathcal{A}$ is the class of all subsets of $L$ that are bounded or compact or absolutely convex and compact or precompact with respect to any one of the weak, Mackey, or strong topologies, then $P'(\mathcal{A})$ is the class of all subsets of $E'$ of the same type. Thus, in particular, if $T$ is the weak, Mackey, or strong topology respectively, then $T_I$ is the weak, Mackey, or strong topology. These results follow from the weak (and thus strong and Mackey) continuity of the maps $I'$ and $P'$ and from the fact that $P'I'$ is the identity on $E'$.

Moreover, each of the above classes satisfies the condition in (2) of Lemma 2.2. Thus, $P$ is $T$, $T_I$ continuous for the corresponding topologies.

If we take $\langle L, L' \rangle = \langle S(E), S'(E') \rangle$, $P = P_I$, and $I = I_I$, then $P_I$ and $I_I$ are weakly continuous. In fact, it follows easily that 
   $$P_I(y) = I_I(y) = (\theta', \theta', \ldots, \theta', \theta, \theta, \ldots)$$
   and 
   $$I_I[(y_n)] = P_I[(y_n)] = y_I$$
   for all $(y_n)$ in $S'(E')$ as the notation anticipates.

From the previous lemma and remark, we get the following proposition and corollary.

**Proposition 2.3** Let $\langle S(E), S'(E') \rangle$ be a dual pair of v.s.s. and let $T$ be the polar topology on $S(E)$ of $\mathcal{A}$-convergence. Then:

1. $T_I$ is the polar topology on $E$ of $P_I(\mathcal{A})$-convergence

2. $P$ is $T$, $T_I$ continuous if and only if the class $\mathcal{A}$ determining $T$ can be chosen so that $I'I_P(A) \in \mathcal{A}$ whenever $A \in \mathcal{A}$.

**Corollary 2.4** Let $\langle S(E), S'(E') \rangle$ be a dual pair of v.s.s. Then for each $j$, $P_I$ is continuous and open and $I_I$ is continuous and closed if $S(E)$ and $E$ both have the weak, the Mackey, or the strong topologies, or if $S(E)$ has the normal topology and $E$ has the weak topology.
3. Convergence Properties

In this section, we shall develop properties of the dual pair \( \langle S(E), S'(E') \rangle \) that are well-known for the scalar case where \( E = E' = K \) and \( S'(E') = S(E)^x \) (see Köthe [6]). The key results are obtained from corresponding properties of the particular case \( \langle l_1, l_\infty \rangle \).

**Proposition 3.1** Let \( \langle S(E), S'(E') \rangle \) be a dual pair of v.s.s. with \( S'(E') \) solid. A sequence of elements in \( S(E) \) is weakly convergent (resp. weakly Cauchy) if and only if it is normally convergent (resp. normally Cauchy).

**Proof.** Since \( S'(E') \) is solid, the normal topology is defined. A normally convergent (resp. normally Cauchy) sequence is weakly convergent (resp. weakly Cauchy) since the normal topology is finer than the weak topology.

To prove the reverse implication for convergent sequences, suppose that \( (x_n^{(i)}) \to (x_n) \) weakly in \( S(E) \) as \( i \to \infty \). Since \( S'(E') \) is solid, we have for all \( (c_n) \) in \( l_\infty \) and \( (y_n) \) in \( S'(E') \) that \( (c_n y_n) \) is in \( S'(E') \) and thus that

\[
\left\langle (x_n^{(i)}) - (x_n), (c_n, y_n) \right\rangle = \left| \sum_{n=1}^{\infty} \left\langle x_n^{(i)} - x_n, y_n \right\rangle c_n \right| \to 0 \text{ as } i \to \infty.
\]

This implies that \( (\langle x_n^{(i)} - x_n, y_n \rangle) \) converges to \( (0) \) weakly in \( l_1 \) as \( i \to \infty \), where the weak topology on \( l_1 \) is defined by the dual pair \( \langle l_1, l_\infty \rangle \). Since weak convergence implies normal convergence in \( l_1 \) ([7], p. 198), the sequence also converges normally. Consequently, \( \sum_{n=1}^{\infty} \left| \langle x_n^{(i)} - x_n, y_n \rangle \right| \to 0 \) as \( i \to \infty \) for each \( (y_n) \) in \( S'(E') \). That is, \( (x_n^{(i)}) \to (x_n) \) normally as \( i \to \infty \).

If there were a weakly Cauchy sequence \( (x_n^{(i)}), i = 1, 2, \ldots \), in \( S(E) \) which was not normally Cauchy, then for some \( \varepsilon > 0 \) and \( (y_n) \) in \( S'(E') \), there would be a sequence of pairs \( i_k, j_k \) where \( k = 0, 1, 2, \ldots \), such that \( \sum_{n=1}^{\infty} \left| \langle x_n^{(i_k)} - x_n^{(j_k)}, y_n \rangle \right| \geq \varepsilon \) contradicting the fact that \( (x_n^{(i)} - x_n^{(j)}) \) converges to \( (0) \) weakly and normally as \( k \to \infty \).

**Remark 3.2** The proposition need not hold if \( S'(E') \) is not solid, even in the case when \( E = E' = K \). For let \( \alpha \) be the scalar sequence space spanned by the finite sequences \((c, -c, c, -c, \ldots) \) where \( c \in K \). If we take \( S(E) = \alpha^x = l_1 \), \( S'(E') = \alpha \), and \( (x_n^{(i)}) = (0, 0, 0, 0, 1, 0, 1, 0, 0, \ldots) \) in \( \alpha^x \), then \( (x_n^{(0)}) \) converges weakly to \( (0) \), but not normally.

For \( (x_n) \) in \( o(E) \), let \( (x_n) (\leq i) = (x_1, x_2, \ldots, x_i, \theta, \theta, \ldots) \), and let

\[
(x_n) (> i) = (x_n) - (x_n) (\leq i).
\]

Given the dual pair \( \langle S(E), S'(E') \rangle \), it is clear that for each \( (x_n) \) in \( S(E) \), \( (x_n) (\leq i) \to (x_n) \) weakly as \( i \to \infty \). Moreover, if the normal topology on \( S(E) \) is defined, then \( S'(E') \subset S(E)^x \) and thus \( (x_n) (\leq i) \to (x_n) \) normally as \( i \to \infty \) for each \( (x_n) \) in \( S(E) \). We now show that if \( S(E) \) is solid, the convergence also holds in the Mackey topology.

**Lemma 3.3** Let \( \langle S(E), S'(E') \rangle \) be a dual pair of v.s.s. with \( S(E) \) solid. Then the map \( (x_n) [-]: S'(E') \to l_1 \) defined by \( (x_n) ([y_n]) = \langle x_n, y_n \rangle \) is \( T_*(S(E), S'(E')) \), \( T_* (l_\infty, l_1) \) continuous.

**Proof.** Since \( S(E) \) is solid, the image of \( (x_n) [-] \) lies in \( l_1 \). Let \( (a_n) \) be in \( l_\infty \). Then \((a_n)^x \) is a typical subbasic weak neighborhood of \((0) \) in \( l_1 \). For \((y_n) \) in the weak neighborhood \( \{(a_n x_n)^x \} \) of \((0) \) in \( S'(E') \) we have \( \left| \sum \langle x_n, y_n \rangle a_n \right| = \left| \sum \langle a_n x_n, y_n \rangle \right| \leq 1 \). Thus, \((x_n) [-] \) takes \((a_n x_n)^x \) into \((a_n)^x \), and continuity follows.

**Proposition 3.4** Let \( \langle S(E), S'(E') \rangle \) be a dual pair of v.s.s. with \( S(E) \) solid. Then for each \( (x_n) \) in \( S(E) \), \( (x_n) (\leq i) \to (x_n) \) in the topology \( T_*(S'(E'), S(E)) \) as \( i \to \infty \).
Proof. Let C be an absolutely convex weakly compact subset of $S'(E')$. By Lemma 3.3, $(x_n) [C]$ is an absolutely convex weakly compact subset of $l_1$. Thus ([6], p. 284),

$$\sup_{(y_n) \in C} \left| \sum_{n=1}^{\infty} \langle x_n, y_n \rangle \right| \to 0 \text{ as } i \to \infty.$$ 

In particular, $\sup_{(y_n) \in C} |\langle(x_n) - (x_n)(\leq i), (y_n)\rangle| \to 0$ as $i \to \infty$, and the proposition follows.

Remark 3.5 1. Proposition 3.4 need not hold if $S(E)$ is not solid. Let $S(E) = \alpha$, $S'(E') = \alpha^*$, and $(\alpha_n) = (1, -1, 1, -1, \ldots)$ where $\alpha$ is defined in Remark 3.2. Because it is orthogonal to the alternating sequences, it turns out that $C = \{(c_1, c_2, c_3, \ldots) : \sum \left| c_n \right| \leq 1\}$ is an absolutely convex, weakly bounded and weakly complete (and therefore weakly compact) subset of $\alpha^*$. However, $(x_n)(\leq i)$ does not converge to $(\alpha_n)$ uniformly on $C$ as $i \to \infty$.

2. We note that if $S(E)$ is solid, then the solid hulls in $S'(E')$ of the sets in the class defining $S(E)$ are weakly bounded by Proposition 5.1 in Section 5. Thus, they determine a polar topology $|T_k|(S'(E'), S(E))$. Since the series occuring in the proof of Proposition 3.4 is absolutely convergent, we can conclude that $(x_n)(\leq i) \to (x_n)$ in the topology $|T_k|(S'(E'), S(E))$ as $i \to \infty$ if $S(E)$ is solid.

If $(S(E), T)$ is an l.c.t.v.s.s., we let $(S(E), T)_a$ be the set of all $(x_n)$ in $(S(E), T)$ for which $(x_n)(\leq i) \to (x_n)$ as $i \to \infty$.

Theorem 3.6 Let $S(E)$ be a solid v.s.s. and let $E' = E^*$ be the algebraic dual of $E$. Then there is a finest l.c.t.v.s.s. topology $T_a$ in $S(E)$ for which $(S(E), T_a) = S(E)$. Moreover, $T_a = T_k(S(E)^*, S(E))$ and $T_a$ is solid.

Proof. If $S(E)$ is solid, then $(S(E), T_k)_a = S(E)$ by Proposition 3.4, where

$$T_k = T_k(S(E)^*, S(E)).$$

Suppose $(S(E), T)_a = S(E)$. Let $f$ be in $(S(E), T)'$. Since $(x_n)(\leq i) \to (x_n)$ in $(S(E), T)$ as $i \to \infty$, we have for each $(x_n)$ in $S(E)$ that

$$f((x_n)) = \lim_{i \to \infty} f((x_n)(\leq i)) = \lim_{i \to \infty} \sum_{n=1}^{i} \langle x_n, y_n \rangle = \langle (x_n), (y_n) \rangle$$

where $y_n = f \circ I_n$ is in $E^*$. Thus, $(S(E), T)' \subset S(E)^*$. Therefore, $T \subset T_k$. Thus, $T_a$ is defined and equals $T_k(S(E)^*, S(E))$.

From the second part of Remark 3.5, it follows that $(S(E), |T_a|)_a = S(E)$. Since $|T_a| \subset T_a$, we have $|T_a| = T_a$. Thus, $T_a$ is solid.

4. Compactness

A net $(x_n^{(a)})$ in an l.c.t.v.s.s. $(S(E), T)$ is called TK-convergent (resp. TK-Cauchy) if for each $j$, $x_j^{(a)}$ is convergent (resp. Cauchy) in $(E, T_j)$. Note that if $P_j$ is $T$, $T_j$ continuous for each $j$, then convergent (resp. Cauchy) nets in $(S(E), T)$ are TK-convergent (resp. TK-Cauchy).

The following theorem is a simple extension of Theorem 2.5 in Pietsch ([8], p. 34).

Theorem 4.1 Let $(S(E), T)$ be an l.c.t.v.s.s. and let $P_j$ be $T$, $T_j$ continuous for each $j$. Then a subset $C$ of $(S(E), T)$ is relatively compact if and only if:

1. For each $j$, $P_j(C)$ is relatively compact in $(E, T_j)$ and
2. Every TK-convergent net in $C$ converges in $(S(E), T)$.

Necessity. Since $P_j$ is $T$, $T_j$ continuous for each $j$, $P_j(C)$ is relatively compact in $(E, T_j)$ if $C$ is relatively compact in $(S(E), T)$.
Let \((x^{(a)}_n)\) be a TK-convergent net in \(C\). Then for each \(j\), there is an \(x_j\) such that \(x^{(a)}_n \to x_j\) in \((E, T_1)\). Since \(C\) is relatively compact, \((x^{(a)}_n)\) has a cluster point in \((S(E), T)\).

Let \(P_j\) be \(T_j\)-continuous and \((x^{(a)}_n)\) is TK-convergent, this cluster point must be \((x_n)\). Thus \((x^{(a)}_n)\) converges in \((S(E), T)\) to the unique cluster point \((x_n)\), since \(C\) is relatively compact.

**Sufficiency.** Let \(\overline{C}\) be the closure of \(C\) in \((S(E), T)\) and let \(\mathcal{F}\) be an ultrafilter to which \(\overline{C}\) belongs. Let \(\overline{P_j}(\overline{C})\) be the closure of \(P_j(C)\) in \((E, T_1)\). Then \(P_j(\mathcal{F})\) is an ultrafilter to which \(\overline{P_j}(\overline{C})\) belongs. For each \(F \in \mathcal{F}\), choose \((a^{(F)}_i)\) in \(F\). Since \(/C/\) is compact, \((a^{(F)}_i)\) converges to \((a)\) in \(S(E, T)\).

Thus \((x^{(a)}_n)\) converges to \((x_n)\) in \((S(E), T)\) to the unique cluster point \((x_n)\), since \(C\) is relatively compact.

Theorem 4.2 Let \((S(E), T)\) be an l.c.t.v.s.s. and let \(P_j\) be \(T_j\)-continuous for each \(j\). Then the solid hull of a subset \(C\) of \(S(E)\) is relatively compact if and only if the following conditions hold:

1. For each \(j\), \(P_j(C)\) is relatively compact in \((E, T_j)\)
2. \((x^{(a)}_n)\) converges to \((a)\) in \((S(E), T)\) uniformly for all \((x_n)\) in \(|C|\) in \((S(E), T)\)
3. \(|C|\) is relatively AK-complete in \((S(E), T)\).

Necessity. Suppose \(|C|\) is relatively compact in \((S(E), T)\). Then \(C\) is also relatively compact, and since \(P_j\) is \(T_j\)-continuous, \(P_j(C)\) is relatively compact in \((E, T_j)\) for each \(j\).

Suppose (2) did not hold. Then for some neighborhood \(U\) of \((a)\) in \((S(E), T)\), there would be a sequence \((x^{(a)}_n)\) in \(|C|\) such that \((x^{(a)}_n)\) is TK-convergent to \((a)\) in \((S(E), T)\). Therefore, by Theorem 4.1, it converges to \((a)\) in \((S(E), T)\). Thus, we obtain a contradiction.

Let \((x_n)\) in \(|C|\) be a Cauchy sequence in the closure of \(|C|\) in \((S(E), T)\). Since \((x_n)\) is TK-convergent and the closure of \(|C|\) is compact, by Theorem 4.1, \((x_n)\) converges to \((a)\) in \((S(E), T)\). We can choose \(\alpha\) such that

\[
(x^{(a)}_n) \to (x_n) \quad \text{as} \quad n \to \infty.
\]

Moreover, using this convergence and (2), we can choose \(\alpha\) such that

\[
(x_n) - (x) = \sum_{i=U/4}^{U/4+1} I_n^{(a)}(x_n) - x_n
\]

for \(\alpha\) sufficiently large. Thus, for each \(i\), \((x^{(a)}_n) \to (x_n)\) in \((S(E), T)\).

Sufficiency. We will use Theorem 4.1 to show that \(|C|\) is relatively compact. Let \((x^{(a)}_n)\) be a net in \(|C|\) such that for each \(j\), \((x^{(a)}_n) \to x_j\) in \((E, T_j)\) and let \(U\) be an absolutely convex neighborhood of \((a)\) in \((S(E), T)\). By the TK-convergence, we have for each fixed \(i\) that

\[
(x^{(a)}_n) \to (x_n) \quad \text{as} \quad n \to \infty.
\]

for \(\alpha\) sufficiently large. Thus, for each \(i\), \((x^{(a)}_n) \to (x_n)\) in \((S(E), T)\). In particular, \((x_n)\) is in the closure of \(|C|\) in \((S(E), T)\). Moreover, using this convergence and (2), we can choose \(\alpha\) such that

\[
(x_n) - (x) = [(x_n) - (x^{(a)}_n)] + [(x^{(a)}_n) - (x)]
\]

for \(\alpha\) sufficiently large. Thus, for each \(i\), \((x^{(a)}_n) \to (x_n)\) in \((S(E), T)\). In particular, \((x_n)\) is in the closure of \(|C|\) in \((S(E), T)\).
for each $i, j$ sufficiently large. That is, $(x_n) (\leq i), i = 1, 2, \ldots$ is a Cauchy sequence in the closure of $|C| \cap S(E)$. By (3), we thus have $(x_n) (\leq i) \to (x_n)$ in $(S(E), T)$ as $i \to \infty$.

From the above convergences and (2), we can choose $i$ such that

$$(x_n^{(a)}) - (x_n) = [(x_n^{(a)}) - (x_n) (\leq i)] + [(x_n^{(a)}) (\leq i) - (x_n) (\leq i)] + [(x_n) (\leq i) - (x_n)]$$

$\in U/3 + U/3 + U/3 = U$

for $\alpha$ sufficiently large. Thus $(x_n^{(a)}) \to (x_n)$ in $(S(E), T)$ and the theorem is proved.

**Remarks.** 1. From (2) we see that if $|C| n \subseteq (E)$ is relatively compact in $(S(E), T)$ then $|C| n \subseteq S(E) \subseteq (S(E), T)$.

2. If $(S(E), T)$ is of Köthe type and $T$ is the polar topology of $\mathcal{A}$-convergence, then condition (2) can also be stated as:

$$\sup_{\{y_n\} \in \mathcal{A}} \sum_{n=1}^{\infty} \left| \langle x_n, y_n \rangle \right| \to 0 \text{ as } i \to \infty \text{ for each } A \in \mathcal{A}.$$ 

Also, if $(S(E), T)$ is AK-complete (in particular, by Proposition 6.1, if $S(E) = S'(E')^*$), then (3) will always hold.

**Proposition 4.3** Let $(S(E), T)$ be of Köthe type and let $T$ be solid. Let $| (x_n) |$ is compact if and only if $(x_n) (\leq i) \to (x_n)$ in $(S(E), T)$ as $i \to \infty$.

**Proof.** The necessity follows immediately from (2) of Theorem 4.2.

To prove the sufficiency, we apply Theorem 4.2. Condition (1) clearly holds. Since $T$ is solid and $(x_n) (\leq i) \to (x_n)$, condition (2) as stated in the remarks above also holds. Since $T$ is solid, $P_j$ is $T$, $T_j$ continuous for each $j$. Therefore, $| (x_n) |$ is closed, since any net in $| (x_n) |$ that converges in $(S(E), T)$ is TK-convergent, and thus has its limit in $| (x_n) |$. Moreover, $| (x_n) |$ is AK-complete in $(S(E), T)$. For if $(z_n) (\leq i), i = 1, 2, \ldots$ is a sequence in $| (x_n) |$, then $(z_n)$ is in $| (x_n) |$; and, if $(z_n) (\leq i), i = 1, 2, \ldots$ is Cauchy then $(z_n) (\leq i) \to (z_n)$ as $i \to \infty$ since $(S(E), T)$ is of Köthe type. Thus (3) holds and $| (x_n) |$ is relatively compact. Being closed, $| (x_n) |$ is compact.

**Corollary 4.4** If $(S(E), S'(E'))$ is a dual pair of v.s.s. with $S'(E')$ solid, then $T_0 (S'(E'), S(E)) \subseteq T_k (S'(E'), S(E))$.

**Proof.** Since $S'(E')$ is solid, the normal topology is defined on $S'(E')$. Also, for each $(y_n) \in S'(E')$, $(y_n) (\leq i) \to (y_n)$ in the normal topology on $S'(E')$ as $i \to \infty$. By Proposition 4.3, $| (y_n) |$ is normally compact; in particular, it is weakly compact. Thus the solid hull of an element of $S'(E')$ is always absolutely convex and weakly compact; that is, $T_0 (S'(E'), S(E)) \subseteq T_k (S'(E'), S(E))$.

## 5. Solid Topologies

**Proposition 5.1** Let $(S(E), S'(E'))$ be a dual pair of v.s.s. If $S(E)$ is solid, then the solid hull in $S'(E')$ of a weakly bounded subset of $S'(E')$ is weakly bounded. Thus, $T_0 (S'(E'), S(E))$ is solid if $S(E)$ is solid.

**Proof.** Let $Y$ be a weakly bounded subset of $S'(E')$. Let $(a_n^{(i)} y_n^{(i)}), i = 1, 2, \ldots$ be any sequence in $| Y | \cap S'(E')$, where $(y_n^{(i)})$ is in $Y$ and $| a_n^{(i)} | \leq 1$ for all $i, n = 1, 2, \ldots$. Let $e^{(i)}, i = 1, 2, \ldots$ be a null sequence in $K$. Since $Y$ is weakly bounded, $e^{(i)} (y_n^{(i)})$ converges weakly to $(\theta')$ as $i \to \infty$. By Proposition 3.1, it converges normally as well. Thus, $e^{(i)} (a_n^{(i)} y_n^{(i)})$ converges normally and weakly to $(\theta')$. Since this convergence holds for all null sequences $e^{(i)}$ and sequences $(a_n^{(i)} y_n^{(i)})$ in $| Y | \cap S'(E')$, it follows that $| Y | \cap S'(E')$ is weakly bounded.
Corollary 5.2 If \( S'(E') \) is solid, then the solid hull of a strongly bounded subset of \( S'(E') \) is strongly bounded. Thus, the polar topology \( T_{b^*}(S'(E'), S(E)) \) of convergence on strongly bounded subsets of \( S'(E') \) is solid if \( S'(E') \) is solid.

**Proof.** Let \( Y \) be a strongly bounded subset of \( S'(E') \) and let \( X \) be a weakly bounded subset of \( S(E) \). Since \( |X| \cap S(E) \) is weakly bounded, there is a positive constant \( M \) such that \( \| \sum (x_n, y_n) \| \leq M \) for all \( (x_n) \) in \( |X| \cap S(E) \), \( (y_n) \) in \( Y \). Because \( S(E) \supseteq \varphi(E) \) and \( |X| \cap S(E) \) is solid in \( S(E) \), this implies that \( \| \sum (x_n, y_n) \| \leq M \) for all \( (x_n) \) in \( X \), \( (y_n) \) in \( Y \). Since such an inequality holds for each weakly bounded \( X \), \( |Y| \) is strongly bounded.

**Remark.** Proposition 5.1 need not hold if \( S(E) \) is not solid. For the sequence \( (x^{(i)}) = (0, 0, \ldots, 1, 1, \ldots, 1, 0, 0, \ldots) \), \( i = 1, 2, \ldots \), is \( T_{a^*}(x, \alpha^x) \) bounded in \( \alpha^x \) (defined in Remark 3.2), but its solid hull is not.

In order to show that \( T_{b^*}(S(E)^*, S(E)) \) is solid when \( S(E) \) is solid, we first prove a lemma that is of interest in itself.

**Lemma 5.3** Let \( \langle S(E), S'(E') \rangle \) be a dual pair of v.s.s. with \( S(E) \) solid and \( S'(E') = S(E)^* \), and let \( T \) be a polar topology on \( S(E) \). Then \( (S(E), T)^{\prime} = S(E)^* \) if and only if \( (S(E), T)_\alpha = S(E) \) and \( (E, T_j)^{\prime} = E' \) for each \( j \).

**Sufficiency.** Since \( T \) is a polar topology, it is enough to show that \( (S(E), T)^{\prime} \subset S(E)^* \). If \( f \) is in \( (S(E), T)^{\prime} \), then since \( (x_n) \leq i \rightarrow (x_n) \) in \( (S(E), T) \) as \( i \rightarrow \infty \), we have \( f[(x_n)] = \lim_{i \rightarrow \infty} f[(x_n)](\leq i) = \lim_{i \rightarrow \infty} \sum_{n=1}^{i} (x_n, y_n) = (x_n, (y_n)) \), where \( y_n = f \circ I_n \) is in \( E' \) since \( f \circ I_n \) is in \( (E, T_j)^{\prime} \). \( (y_n) \) is in \( S(E)^* \) since the series converges for all \( (x_n) \) in \( S(E) \).

**Necessity.** Since \( T \subseteq T_k(S(E)^*, S(E)) \), the weakly bounded subsets of \( S(E)^* \) determining \( T \) can be assumed to be absolutely convex and weakly compact. Since \( P^j \) is weakly continuous, it follows that the weakly bounded subsets of \( E' \) determining \( T_j \) can also be assumed to be absolutely convex and weakly compact. Thus, \( T_j \supseteq T_k(E', E) \); equivalently, \( (E, T_j)^{\prime} = E' \). \( (S(E), T)_\alpha = S(E) \) by Proposition 3.4.

**Proposition 5.4** \( T_k(S(E)^*, S(E)) \) is solid if \( S(E) \) is solid.

**Proof.** By the second part of Remark 3.5, \( (S(E), |T_k|)_\alpha = S(E) \). Clearly

\[ T_k(E', E) = |T_k|_\alpha. \]

Thus, \( |T_k| \subset T_k \) by Lemma 5.3. Since \( |T_k| \supset T_k \) always holds, we have equality and the proposition follows.

6. Completeness and the Space \( S_\mathcal{A}(E) \)

If \( \langle S(E), S'(E') \rangle \) is a dual pair of v.s.s. and \( T \) is the polar topology on \( S(E) \) of \( \mathcal{A} \)-convergence, we let

\[ S_\mathcal{A}(E) = \{ (x_n) \in o(E) : \sup_{\{y_n\} \in \mathcal{A}} \| \sum (x_n, y_n) \| < \infty \text{ for all } A \in \mathcal{A} \}. \]

Then \( S(E) \subset S_\mathcal{A}(E) \subset S'(E')^* \). Moreover, \( \langle S_\mathcal{A}(E), S'(E') \rangle \) is a dual pair of v.s.s. and \( \mathcal{A} \) is a class of weakly bounded subsets of \( S'(E') \). We let \( T_\mathcal{A} \) be the polar topology on \( S_\mathcal{A}(E) \) determined by the class \( \mathcal{A} \). Note that \( T \) is the restriction of \( T_\mathcal{A} \) to \( S(E) \). If \( T \) is solid, \( S_\mathcal{A}(E) \) and \( T_\mathcal{A} \) are independent of the class \( \mathcal{A} \) determining \( T \). Spaces of the type \( (S_\mathcal{A}(E), T_\mathcal{A}) \) with \( T \) solid and \( E = E' = K \) have been considered by Garling [2].

In the following two propositions, we obtain results that will be used in studying the completeness of \( (S(E), T) \).
Proposition 6.1 Let $T = T(S'(E'), S(E))$ be the polar topology on $S(E)$ of $\mathscr{A}$-convergence. Then $(S(E), T)$ is AK-complete if and only if $S(E) \supseteq (S_{\mathscr{A}}(E), T_{\mathscr{A}})$.  

Necessity. If $(x_n)$ is in $(S_{\mathscr{A}}(E), T_{\mathscr{A}})$ then $(x_n) (\leq i)$, $i = 1, 2, \ldots$, is Cauchy in $(S_{\mathscr{A}}(E), T_{\mathscr{A}})$, and thus Cauchy in $(S(E), T)$. If $(S(E), T)$ is AK-complete, it then follows that $(x_n)$ is in $S(E)$.  

 Sufficiency. Note that $(S_{\mathscr{A}}(E), T_{\mathscr{A}})$ is AK-complete. For if $(x_n) (\leq i)$, $i = 1, 2, \ldots$, is Cauchy in $(S_{\mathscr{A}}(E), T_{\mathscr{A}})$, then for each $A$ in $\mathscr{A}$, $|\sum_{n=1}^{\infty} \langle x_n, y_n \rangle | = |\langle (x_n) (\leq i), (y_n) \rangle |$ is bounded uniformly for $(y_n)$ in $A$ and $i = 1, 2, \ldots$. Thus, since $\Sigma \langle x_n, y_n \rangle$ converges, sup $|\sum_{n=1}^{\infty} \langle x_n, y_n \rangle | < \infty$; that is, $(x_n)$ is in $S_{\mathscr{A}}(E)$. Now suppose that $S(E) \supseteq (S_{\mathscr{A}}(E), T_{\mathscr{A}})$ and $(x_n) (\leq i)$, $i = 1, 2, \ldots$, is Cauchy in $(S(E), T)$. Then it is also Cauchy in $(S_{\mathscr{A}}(E), T_{\mathscr{A}})$. Thus, $(x_n) (\leq i) \rightarrow (x_n)$ in $(S_{\mathscr{A}}(E), T_{\mathscr{A}})$ as $i \rightarrow \infty$, since $(S_{\mathscr{A}}(E), T_{\mathscr{A}})$ is AK-complete. Therefore, $(x_n)$ is in $(S_{\mathscr{A}}(E), T_{\mathscr{A}})$, and thus in $S(E)$.  

Proposition 6.2 Let $(S(E), T)$ be of Köthe type, where $T$ is a solid polar topology of $\mathscr{A}$-convergence. Then each of the following statements implies the one below it:  

(1) $S(E) = S'(E')^*$  
(1') $S(E)$ is a closed subspace of $(S_{\mathscr{A}}(E), T_{\mathscr{A}})$  
(2) Every TK-convergent Cauchy net in $(S(E), T)$ converges  
(2') Every TK-convergent bounded Cauchy net in $(S(E), T)$ converges  
(2'') Every TK-convergent Cauchy sequence in $(S(E), T)$ converges  
(3) $(S(E), T)$ is AK-complete.  

Moreover, (1') and (2) are equivalent. Also, if $T$ is the normal topology, then all the above statements are equivalent.  

Proof. If (1) holds, then $S(E) = S'(E')^* = S_{\mathscr{A}}(E)$ and thus (1') holds.  

Suppose (1') holds and $(x_n^{(a)})$ is a Cauchy net in $(S(E), T)$ such that for each $j$, $x_n^{(a)} \rightarrow x_j$ in $(E, T_j)$. Let $\varepsilon > 0$ and $A$ in $\mathscr{A}$ be given. Then, 

$\sum_{n=1}^{\infty} |\langle x_n^{(a)}, y_n \rangle | \leq \sum_{n=1}^{\infty} |\langle x_n^{(a)} - x_n^{(b)}, y_n \rangle | + \sum_{n=1}^{\infty} |\langle x_n^{(b)} - x_n, y_n \rangle |$. 

Since $(x_n^{(b)})$ is Cauchy in $(S(E), T)$, there is some $\gamma$ such that $\alpha, \beta \geq \gamma$ implies that $\sum_{n=1}^{\infty} |\langle x_n^{(a)} - x_n^{(b)}, y_n \rangle | \leq \varepsilon/2$ for all $(y_n)$ in $A$. Since for each $j$, $x_n^{(a)} \rightarrow x_j$ in the topology $T_j$ of $\mathscr{A}$-convergence, for sufficiently large $\beta$, $\sum_{n=1}^{\infty} |\langle x_n^{(b)} - x_n, y_n \rangle | \leq \varepsilon/2$ for all $(y_n)$ in $A$. Since the choice of $\gamma$ was independent of $k$, we have that $\alpha \geq \gamma$ implies that 

$\sum_{n=1}^{\infty} |\langle x_n^{(a)} - x_n, y_n \rangle | \leq \varepsilon$ 

for all $(y_n)$ in $A$. In particular, it follows that $(x_n)$ is in $S_{\mathscr{A}}(E)$ since $(x_n^{(a)}) \rightarrow (x_n)$ and $(x_n^{(a)})$ are. Also, $(x_n^{(a)}) \rightarrow (x_n)$ in $(S_{\mathscr{A}}(E), T_{\mathscr{A}})$. Since we are assuming (1'), this implies that $(x_n^{(a)}) \rightarrow (x_n)$ in $(S(E), T)$. Thus (1') implies (2).  

The remaining implications follow immediately.  

If $T$ is the normal topology, then $(x_n) (\leq i)$ is normally Cauchy for each $(x_n)$ in $S'(E')^*$. Thus, (3) implies (1) if $T$ is the normal topology.  

Remark 6.3 Recall that if $S'(E')$ is solid, then weakly convergent (weakly Cauchy) sequences are normally convergent (normally Cauchy). Thus if $S'(E')$ is solid, (2'') and (3) are equivalent to the condition $S(E) = S'(E')^*$ when $T$ is the weak topology.  

Suppose that the mapping $I$ discussed in Section 2 is $T_1$, $T$ closed. Then $I(E)$ is a closed subspace of $(L, T)$. Consequently, $(E, T_1)$ is complete (resp. quasicomplete, sequentially complete) if $(L, T)$ is complete (resp. quasicomplete, sequentially complete). From these facts and the preceding proposition and remark, we obtain the following theorems.
Theorem 6.4 Let \((S(E), T)\) be of Köthe type, and let \(T\) be solid. Then
1. \((S(E), T)\) is complete if and only if \((E, T_j)\) is complete for all \(j\) and \(S(E)\) is a closed subspace of \((S_\sigma(E), T_\sigma)\).
2. If \(S(E)\) is a closed subspace of \((S_\sigma(E), T_\sigma)\) then \((S(E), T)\) is quasicomplete (resp. sequentially complete) if and only if \((E, T_j)\) is quasicomplete (resp. sequentially complete) for all \(j\).

Theorem 6.5
1. \(S(E)\) is normally complete (resp. quasicomplete, sequentially complete) if and only if \((E, T_j)\) is normally complete (resp. quasicomplete, sequentially complete) and \(S(E) = S'(E')^*\).
2. If \(S'(E')\) is solid, then \(S(E)\) is weakly sequentially complete if and only if \((E, T_j)\) is weakly sequentially complete and \(S(E) = S'(E')^*\).

Remark. If \(T\) is solid and \(S(E) = S'(E')^*\), then \(S(E) = S_\sigma(E)\) is a closed subspace of \((S_\sigma(E), T_\sigma)\) and (1) and (2) of Theorem 6.4 apply. Another example of a closed subspace is provided by the following proposition.

Proposition 6.6 If \((S(E), T)\) is an l.c.t.v.s.s. with \(T\) solid, then \((S(E), T)\) is the closure of \(\varphi(E)\) in \((S(E), T)\). In particular, \((S(E), T)\) is a closed subspace.

Proof. Clearly \((S(E), T)\) is contained in the closure of \(\varphi(E)\). Suppose then that \((x_n)\) is in the closure of \(\varphi(E)\) and \(U\) is a solid neighborhood of \((\theta)\) in \((S(E), T)\). We wish to show that \((x_n) (\geq i) = (x_n) - (z_n) (\leq i)\) is in \(U\) for all \(i\) sufficiently large. Choose \((z_n)\) in \(\varphi(E)\) such that \((x_n) - (z_n)\) is in \(U\). Since \(U\) is solid, \((x_n) (\geq i) - (z_n) (\geq i)\) is in \(U\) for all \(i\). Thus, \((x_n) (\geq i)\) is in \(U\) for all \(i\) sufficiently large since \((z_n)\) is in \(\varphi(E)\).

Corollary 6.7 If \((S(E), T)\) is of Köthe type, where \(T\) is a solid polar topology of \(\mathcal{K}\)-convergence, then \((S_\sigma(E), T_\sigma)\) is a closed subspace of \((S_\sigma(E), T_\sigma)\).

Corollary 6.8 Let \((S(E), T)\) be of Köthe type with \(T\) solid and \((S(E), T) = S(E)\). Then \((S(E), T)\) is complete if and only if \((E, T_j)\) is complete for all \(j\) and \((S(E), T)\) is AK-complete.

Proof. The necessity is immediate.

To prove the sufficiency, suppose that \(T\) is the polar topology of \(\mathcal{A}\)-convergence. If \((S(E), T)\) is AK-complete and \((S(E), T) = S(E)\), then \(S(E) = (S_\sigma(E), T_\sigma)\) by Proposition 6.1. Thus, \(S(E)\) is a closed subspace of \((S_\sigma(E), T_\sigma)\) by Corollary 6.7. Therefore, \((S(E), T)\) is complete.

Remarks. 1. By Proposition 3.4, if \(S(E)\) is solid, the condition \((S(E), T) = S(E)\) of Corollary 6.8 holds for any polar topology \(T\) coarser than \(T_\sigma(S'(E'), S(E))\).

2. A v.s.s. need not be an \(\alpha\)-dual to be complete, even if the topology is coarser than the Mackey topology. For example, the scalar space \(c_0\) of null sequences is complete with respect to the topology \(T_\sigma(l_\infty, l_1)\) defined by the norm \(\|c_n\| = \sup |c_n|\).

3. A perfect v.s.s. need not be weakly complete, or even weakly quasicomplete. \(l_1\) is not \(T_\sigma(l_\infty, l_1)\) quasicomplete.

7. Matrix Representation of Linear Maps

In this section, we consider two dual pairs \(\langle E, E' \rangle\) and \(\langle F, F' \rangle\) of vector spaces and two dual pairs \(\langle S(E), S'(E') \rangle\) and \(\langle S(F), S'(F') \rangle\) of v.s.s. The projection mappings and the bilinear forms for these spaces will be distinguished by subscripts.

A matrix \((Z_{ij})\), \(i, j = 1, 2, \ldots\), of linear maps \(Z_{ij}: E \to F\) is said to represent the linear map \(Z: S(E) \to S(F)\) if for all \((x_n)\) in \(S(E)\), \(\sum_{i=1}^{\infty} Z_{ij} x_j\) is a weakly convergent sum in \(F\) for each \(i\) and \(P_{i,F}(Z(x_n)) = \sum_{j=1}^{\infty} Z_{ij} x_j\) for all \(i\).
**Theorem 7.1** Let $S'(E')$ be weakly sequentially complete and let $S'(E') = S(E)^*$. Then a linear map $Z$ from $S(E)$ to $S(F)$ is weakly continuous if and only if it is represented by a matrix of weakly continuous linear maps from $E$ to $F$.

**Proof.** The necessity follows by a straightforward application of the weak continuity of the maps $P_{i,F}$ and $I_{f,E}$ and the weak convergence of $(x_n) (\subseteq i)$ to $(x_n)$ for each $(x_n)$ in $S(E)$.

To obtain the sufficiency, it is enough to show that if $Z$ is represented by a matrix $(Z_{ij})$, where the $Z_{ij}: E \to F$ are weakly continuous, then $Z$ has an adjoint. For all $(x_n)$ in $S(E)$ and $(y_n)$ in $S'(F')$ we have

$$
\langle Z(x_n), (y_n) \rangle_{S(F), S'(F')} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \langle Z_{ij} x_j, y_i \rangle_{F, F'} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \langle Z_{ij} x_j, y_i \rangle_{F, F'}
$$

(since the sum is weakly convergent)

$$
= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \langle x_j, Z_{ij} y_i \rangle_{E, F'}
$$

where $Z_{ij}^*$, the adjoint of $Z_{ij}$, is defined since $Z_{ij}$ is assumed weakly continuous. In particular, this implies that $(Z_{ij}^* y_i)$ is in $S(E)^*$ for each $i$. Therefore it is in $S'(E') = S(E)^*$. Thus,

$$
\langle Z(x_n), (y_n) \rangle_{S(F), S'(F')} = \sum_{i=1}^{\infty} \langle x_n, Z_{in}^* y_i \rangle_{S(E), S'(E')} = \sum_{i=1}^{\infty} \langle x_n, Z_{in}^* y_i \rangle_{S(E), S'(E')}
$$

Since this sum converges for all $(x_n)$ in $S(E)$, we get that $\sum_{i=1}^{\infty} (Z_{in}^* y_i)$ is a weak Cauchy sequence of partial sums of elements of $S'(E')$. By the assumption of weak sequential completeness, it converges to an element $\sum_{i=1}^{\infty} (Z_{in}^* y_i)$ say, of $S'(E')$. Since the convergence is weak, from the previous equality we have

$$
\langle Z(x_n), (y_n) \rangle_{S(F), S'(F')} = \langle x_n, \sum_{i=1}^{\infty} (Z_{in}^* y_i) \rangle_{S(E), S'(E')}
$$

Thus the adjoint of $Z$ is defined and $Z$ is weakly continuous.

**Remark.** From (2) of Theorem 6.5, we see that the conditions of Theorem 7.1 are satisfied if $J_E'$ is weakly sequentially complete, $S(E)$ is solid, and $S'(E') = S(E)^*$. 

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**8. The Vector Sequence Space $\lambda(E, T)$**

We assume throughout this section that $\lambda$ is a solid scalar sequence space, $E$ is a member of the dual pair $\langle E, E' \rangle$ and $T$ is the polar topology on $E$ of $\mathcal{A}$-convergence. We let

$$
\lambda(E, T) = \{(x_n) \in \omega(E) : | \langle x_n, A \rangle | \leq \lambda \text{ for all } A \in \mathcal{A}\}
$$

where $| \langle x_n, A \rangle | = \sup_{y \in A} | \langle x_n, y \rangle |$. Since $\lambda$ is solid, $\lambda(E, T)$ is a v.s.s. and is solid. In particular, $c_0(E, T)$ is the v.s.s. of all sequences in $(E, T)$ converging to the zero vector, and $l_{\infty}(E, T)$ is the v.s.s. of all bounded sequences in $(E, T)$. If $T = T_s(E', E)$ and $\lambda$ is perfect, then $\lambda(E, T) = \lambda(E)$ is the generalized perfect space studied by Pietsch [8].

It is not in general true that $\lambda(E, T)^* = \lambda^*(E', T')$ for some polar topology $T' = T_s(E', E')$ on $E'$. However, if we let $\xi(E', \mathcal{A}) = \{(y_n) \in \omega(E') : y_n \in A \text{ for all } n \text{ for some } A \in \mathcal{A}\}$, and $T^*$ be the polar topology on $E'$ of convergence on the class of bounded subsets of $(E, T)$, then the following proposition holds.

**Proposition 8.1**

1. $(\lambda(E, T))^* = \lambda^*(E', T^*)$

2. $\lambda^{**}(E, T) = [\lambda^* \circ \xi(E', \mathcal{A})]^*$ where

$$
\lambda^* \circ \xi(E', \mathcal{A}) = \{(c_n y_n) : (c_n) \in \lambda^*, (y_n) \in \xi(E', \mathcal{A})\}.
$$

**Proof.** (1) If $(y_n)$ is in $\lambda(E, T)^*$, then $\sum | \langle x_n, y_n \rangle | < \infty$ for all $(x_n)$ in $\lambda(E, T)$. In particular, $\sum | \langle c_n z_n, y_n \rangle | < \infty$ for all $(c_n)$ in $\lambda$ and $(z_n)$ in $l_{\infty}(E, T)$. Thus,
$\Sigma | c_n | < B, y_n > | < \infty$ for each bounded set $B$ in $(E, T)$ and each sequence $(c_n)$ in $\lambda$; otherwise, there would be a sequence $(z_n)$ in $l_\infty(E, T)$ such that $\Sigma | c_n | < < z_n, y_n > |$ diverged. Thus, $(| B, y_n |)$ is in $\lambda^*$ for each bounded set $B$ in $(E, T)$. Therefore, $(y_n)$ is in $\lambda^*(E', T^*)$.

(2) Recalling the appropriate definitions, we have:

$\sum c_n | < x_n, A > | < \infty$ for all $A \in \mathcal{A}$

and

$\sum c_n | < x_n, y_n > | < \infty$ for all $(y_n) \in \xi(E', \mathcal{A})$

Corollary 8.2 If $\lambda^* = l_1$ then $\lambda(E, T)^* = l_1(E', T^*)$.

Proof. Since $\lambda^* = l_1$, $\lambda \subset \lambda^{**} = l_\infty$. Thus, $\lambda(E, T) \subset l_\infty(E, T)$. Therefore $\lambda(E, T)^* \supset l_\infty(E, T)^* \supset l_1(E', T^*)$, where the last inclusion follows immediately from the definitions. The reverse inclusion holds by (1) of Proposition 8.1.

Corollary 8.3 If $E$ is a normed space with norm topology $\| \|$, and $(E, \| \|)^' = E'$ is given the strong norm topology $\| \|'$, then $\lambda(E, \| \|')^* = \lambda^*(E', \| \|')$.

Proof. If $(x_n)$ is in $\lambda(E, \| \|)$ and $(y_n)$ is in $\lambda^*(E', \| \|')$, then

$\Sigma | < x_n, y_n > | \leq \Sigma | x_n \| \| y_n \| < \infty$ since $| x_n \|$ is in $\lambda$ and $| y_n \|$ is in $\lambda^*$.

Thus, $\lambda(E, \| \|')^* \supset \lambda^*(E', \| \|')$. The reverse inclusion follows from (1) of Proposition 8.1.

Remarks. By (2) of Proposition 8.1, it follows that if $\lambda$ is perfect, then $\lambda(E, T)$ is an $\alpha$-dual and is thus perfect. In particular, the generalized perfect spaces $\lambda(E)$ of Pietsch [8] are perfect.

If we take $S(E) = \lambda(E)$ and $S'(E')$ to be the linear hull of $\lambda^*. \xi(E', \mathcal{A})$ where $\mathcal{A}$ is the class of all singletons in $E'$, then the topologies considered by Pietsch [8] are polar topologies of the dual pair $\langle S(E), S'(E') \rangle$. Note, in this case, that $S(E) = S'(E')^*$ by (2) of Proposition 8.1, and that $S(E)$ and $S'(E')$ are solid.

References


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