ABSTRACT

Let $\mathcal{T}_n$ denote the set of irreducible $n \times n$ tournament matrices. Here are our main results: (1) For all $n \geq 3$, every matrix in $\mathcal{T}_n$ has at least three distinct eigenvalues; such a matrix has exactly three distinct eigenvalues if and only if it is a Hadamard tournament matrix. (2) For all $n \geq 3$ there is a matrix in $\mathcal{T}_n$ having $n$ distinct eigenvalues. (3) If $\alpha_n$ denotes the maximum algebraic multiplicity of 0 as an eigenvalue of the matrices in $\mathcal{T}_n$, then $\lfloor n/2 \rfloor - 2 \leq \alpha_n \leq n - 6$ for all $n \geq 8$. Each algebraic multiplicity $m$ with $1 \leq m \leq \lfloor n/2 \rfloor - 2$ is achieved for the eigenvalue 0 by some matrix in $\mathcal{T}_n$ for every $n \geq 6$. (4) If $\pi_n$ is the minimum Perron value (i.e. spectral radius) of all matrices in $\mathcal{T}_n$, then $2 < \pi_n < 2.5$ for all $n \geq 8$.

1. INTRODUCTION

A tournament matrix of order $n$ is an $n \times n$ $(0,1)$ matrix $M$ satisfying $M + M^t = J - I$, where $J$ is the $n \times n$ matrix of all ones. It has been observed recently that the rank of a tournament matrix of order $n$ is at least
n - 1 (see de Caen and Hoffman [9]). Subsequently, Maybee and Pullman [15] showed that if $\lambda$ is an eigenvalue of an $n \times n$ tournament matrix $M$, then $\text{rank}(M - \lambda I) = n - 1$ or $\text{Re}\lambda = -\frac{1}{2}$. Thus the geometric multiplicity of an eigenvalue is one except when its real part is $-\frac{1}{2}$; in this case, we show in Section 2 that the algebraic and geometric multiplicities coincide. What can be said about the algebraic multiplicities of the eigenvalues of a tournament matrix? That question is the subject of our paper.

Let $T_n$ be the $n \times n$ $(0,1)$ matrix $[t_{ij}]$ having $t_{ij} = 1$ if and only if $i < j$; this is a transitive $n \times n$ tournament matrix. It is reducible when $n > 1$ and clearly has $\lambda = 0$ as an eigenvalue of algebraic multiplicity $n$ and geometric multiplicity 1. Other reducible tournament matrices can easily be constructed having eigenvalues of various algebraic multiplicities. Therefore we confine ourselves to the irreducible case.

In Section 2, we note that irreducible tournament matrices of order $n \geq 3$ must have at least three distinct eigenvalues. In Section 3 we characterize those having exactly three; specifically, in Theorem 3.2 we show that for $n \geq 3$, such matrices are precisely the $n \times n$ Hadamard tournament matrices—namely, the tournament matrices $H$ of order $n \equiv 3 \pmod{4}$ satisfying $HH^t = [(n + 1)/4]I + [(n - 3)/4]J$. Hadamard tournament matrices of order $n$ are known to be coexistent with the skew Hadamard matrices of order $n + 1$. Such matrices exist for infinitely many $n$.

In contrast to Section 3, we study in Section 4 a sequence of $n \times n$ irreducible tournament matrices $U_n$ having $n$ distinct eigenvalues. The matrix $U_n$ is obtained from the transitive tournament $T_n$ by exchanging the entries in the $(1,n)$ and $(n,1)$ positions. In Theorem 4.1 we compute the characteristic polynomial of $U_n$ and deduce in Theorem 4.2 that $U_n$ has $n$ distinct eigenvalues. The $U_n$ have the interesting property that they have only one real eigenvalue, the Perron value (also called the Perron root) when $n$ is odd, and only two real eigenvalues (the Perron value and a negative one) when $n$ is even. We calculate the eigenvectors of $U_n$ as functions of its eigenvalues, and show that no eigenvector has a zero entry.

The result of Section 3 shows that for infinitely many $n$, the algebraic multiplicity of a nonreal eigenvalue of an irreducible tournament matrix of order $n$ can be $(n - 1)/2$, the maximum possible for nonreal eigenvalues. How large can the algebraic multiplicity of a real eigenvalue be? In Section 5, we study this question by presenting, for each $n \geq 6$ and each $1 \leq m \leq \lfloor n/2 \rfloor - 2$, irreducible tournament matrices of order $n$ having 0 as an eigenvalue of multiplicity $m$ (Theorem 5.3). Letting $\alpha_n$ denote the maximum algebraic multiplicity of 0 as an eigenvalue of an $n \times n$ irreducible tournament matrix, the previous result shows that $\lfloor n/2 \rfloor - 2 \leq \alpha_n$ for $n \geq 6$. Further, we show in Corollary 5.1.1 that $\alpha_n \leq n - 6$ for all $n \geq 8$, by investigating the minimum Perron value, $\pi_n$, of the irreducible tournament matrices of
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order \( n \). Brauer and Gentry [3] showed that \( \sqrt{2} \) is a lower bound on \( \pi_n \) for all \( n \geq 4 \). We show that, in fact, \( 2 < \pi_n < 2.5 \) for all \( n \geq 8 \) (see Theorem 5.1).

Throughout this paper, we will employ some of the terminology and results of both nonnegative matrix theory and graph theory. The relevant background information on these topics can be found in Horn and Johnson [11] and in Bondy and Murty [1], respectively.

2. GENERAL REMARKS

In what follows, we will use some basic facts concerning the eigenvalues of tournament matrices. These results can be found in [2] and [17], but for the sake of completeness, we will sketch the proofs here. Given an \( n \times n \) tournament matrix \( M \), let \( x \) be an eigenvector of \( M \) corresponding to the eigenvalue \( \lambda \). Pre- and postmultiplying the equation \( M + M^* = J - I \) by \( x^* \) and \( x \) respectively yields \( (2 \Re \lambda + 1)x^*x = x^*j^jx \), where \( j \) is the all ones vector. Thus we have \( \Re \lambda \geq -\frac{1}{2} \) (see [2]), with equality holding if and only if \( x^*j = 0 \) (see [17]). Applying the Cauchy-Schwarz inequality to \( x^*j^jx \) gives \( (2 \Re \lambda + 1)x^*x = x^*j^jx \leq j^jx^*x = nx^*x \), with equality holding if and only if \( x \) is a scalar multiple of \( j \). It follows that the Perron value of a tournament matrix is at most \((n - 1)/2\) (see [2]), with equality holding if and only if the tournament matrix is regular, that is, each of its row sums is \((n - 1)/2\) (see [17]).

It has been shown by Maybee and Pullman [15] that if \( \lambda \) is an eigenvalue of a tournament matrix, then its geometric multiplicity is one whenever \( \Re \lambda \neq -\frac{1}{2} \). On the other hand, given a tournament matrix \( M \) with eigenvalue \( \lambda \) and \( \Re \lambda = -\frac{1}{2} \), the following argument shows that the geometric and algebraic multiplicities of \( \lambda \) are the same.

If the two multiplicities are different, then there is an eigenvector \( x \) corresponding to \( \lambda \) and a generalized eigenvector \( y \) satisfying \( My = \lambda y + x \). Then \( x^*My + x^*M^*y = x^*Jy - x^*y \). Since \( x^*J = 0 \), we have \( x^*(\lambda y + x) + \lambda x^*y = -x^*y \), which yields \( x^*x = 0 \), a contradiction. Thus no such \( y \) can exist, so that the algebraic and geometric multiplicities of \( \lambda \) must coincide.

Next we note that if \( M \) is an irreducible tournament matrix of order \( n \geq 3 \), then \( M \) has at least three distinct eigenvalues. This follows from the fact that the trace of \( M^2 \) is zero, so not all of the eigenvalues of \( M \) can be real. Thus \( M \) has at least one conjugate pair of nonreal eigenvalues, as well as a Perron value, for a total of at least three distinct eigenvalues. Evidently the algebraic multiplicity of a nonreal eigenvalue of an \( n \times n \) tournament
matrix is at most $(n - 1)/2$. In the next section, we characterize the class of such matrices that achieve equality.

3. TOURNAMENT MATRICES WITH THREE DISTINCT EIGENVALUES

A tournament matrix $H$ of order $n$ is a Hadamard tournament matrix if it satisfies the equation $HH^t = [(n + 1)/4]I + [(n - 3)/4]J$. Note that necessarily $n = 3 \pmod{4}$. We remark that the question of the existence of such matrices is a difficult one, since Hadamard tournament matrices of order $n$ are known to be coexistent with skew Hadamard matrices of order $n + 1$. See Geramita and Seberry [10] for a discussion.

**Proposition 3.1.** Let $H$ be a Hadamard tournament matrix of order $n$. Then $H$ has eigenvalues $(n - 1)/2$ (with algebraic multiplicity one) and $-1/2 \pm i\sqrt{n}/2$ (each with algebraic multiplicity $(n - 1)/2$).

**Proof.** Let $p(\lambda) = \det(\lambda I - H)$ and $c = (n - 3)/4$. Then $p^2(\lambda) = \det(\lambda I - H)\det(\lambda I - H^t) = \det(\lambda^2 I - \lambda(H + H^t) + HH^t) = \det((c - \lambda)J + (\lambda^2 + \lambda + c + 1)I) = [(\lambda^2 + \lambda + c + 1 + n(c - \lambda))(\lambda^2 + \lambda + c + 1)^{n-1}]$, the last equality following from the general fact that $\det(aI + bJ) = (a + nb)a^{n-1}$. Thus we have

$$p^2(\lambda) = \left(\lambda - \frac{n - 1}{2}\right)^2 \left(\lambda^2 + \lambda + \frac{n + 1}{4}\right)^{n-1},$$

so that

$$p(\lambda) = \left(\lambda - \frac{n - 1}{2}\right) \left(\lambda^2 + \lambda + \frac{n + 1}{4}\right)^{(n-1)/2}.$$  

Having seen that Hadamard tournament matrices have exactly three distinct eigenvalues (and hence their nonreal eigenvalues have maximum algebraic multiplicity), the question arises whether these are the only such irreducible tournament matrices. Our next result will help us to answer this question in the affirmative.
THEOREM 3.1. Suppose that $M$ is an irreducible nonnegative matrix of order $n > 4$, with integer entries and Perron value $\rho$. Further suppose that the trace of $M^2$ is zero and that $M$ has exactly three distinct eigenvalues. Then $n$ is odd, $\rho$ is an integer, and $(n-1)/2$ divides $\rho$.

Proof. Let the eigenvalues of $M$ be $\rho$, $\lambda_1$, and $\lambda_2$. Since the trace of $M^2$ is zero, $\lambda_1$ and $\lambda_2$ must be a complex conjugate pair, say $\alpha \pm i\beta$, with $\beta \neq 0$. Further, $\rho$ is an algebraically simple eigenvalue, so $\lambda_1$ and $\lambda_2$ must both have algebraic multiplicity $(n-1)/2$; in particular, $n$ is odd. The trace of $M^2$ is zero. Since $M$ is nonnegative, its diagonal entries are zero, so the trace of $M$ is also zero. It follows from these two facts that $\alpha = -\rho/(n-1)$ and $\beta = \rho\sqrt{n}/(n-1)$, so that the eigenvalues of $M$ are $\rho$ and $\rho(-1+i\beta)/(n-1)$.

Note that the traces of $M^3$ and $M^4$ are both integers. Since $[\rho/(n-1)]^3(-1+i\sqrt{n})^3 = \rho^3[-1+3n+i\sqrt{n}(3-n)]/(n-1)^3$, we see that the trace of $M^3$ is $\rho^3+(n-1)\rho^3(-1+3n)/(n-1)^3 = \rho^3n(n+1)/(n-1)^3 = i_1 \in \mathbb{N}$. Also $[\rho/(n-1)]^4(-1+i\sqrt{n})^4 = \rho^4[1-6n+n^2+i\sqrt{n}(4n-4)]/(n-1)^4$, and hence the trace of $M^4$ is $\rho^4+(n-1)\rho^4(1-6n+n^2)/(n-1)^4 = \rho^4n(n-3)(n+1)/(n-1)^3 = i_2 \in \mathbb{N}$. Taking quotients, we find that $i_2/i_1 = \rho/(n-3)/(n-1)$, so that $\rho$ is a rational number. Since $\rho$ is a rational root of $\det(\lambda I - M)$, which is a monic polynomial with integer coefficients, $\rho$ must be an integer. Since $\lambda_1$ and $\lambda_2$ are algebraic integers, so is $\lambda_1 + \lambda_2 = -2\rho/(n-1)$. It follows that $-2\rho/(n-1)$ must be an integer, so that $(n-1)/2$ divides $\rho$.

THEOREM 3.2. Suppose that $M$ is an irreducible tournament matrix of order $n \geq 3$. Then $M$ has exactly three eigenvalues if and only if $M$ is a Hadamard tournament matrix.

Proof. The sufficiency is just Proposition 3.1. To see the other implication, we may assume that $n > 3$, since

$$
\begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
$$

and its transpose are the only irreducible $3 \times 3$ tournament matrices, and both are Hadamard tournament matrices. Consider an irreducible tournament matrix $M$ of order $n \geq 4$ with three distinct eigenvalues. Since Theorem 3.1 applies to $M$, its Perron value is at least $(n-1)/2$. But the Perron value of any tournament matrix of order $n$ is at most $(n-1)/2$, so the
maximum eigenvalue of $M$ is $(n - 1)/2$, and hence $M$ is a regular tournament matrix; i.e., $MJ = JM = [(n - 1)/2]J$. Note that $M$ is a normal matrix, since $MM^t = MJ - M - M^2 = JM - M - M^2 = M'M$. From the proof of Theorem 3.1, we see that the eigenvalues of $M$ are $(n - 1)/2$ and $-i\sqrt{n}/2$.

Since $M$ is normal, there is an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of $M$, say $j$ (which is a Perron vector) and $w_k, k = 1, 2, \ldots, n - 1$. As noted above, $MM^t - MJ - M - M^2 = [(n - 1)/2]J - M - M^2$. Thus $MM^t j = [(n - 1)/2]j - Mj - M^2j = [(n - 1)^2/4]j = [((n - 3)/4]J + [(n + 1)/4]I)j$. Since $j^t w_k = 0$ for $1 \leq k \leq n - 1$, we have $MM^t w_k = MJ w_k - M w_k = [((n - 3)/4]J + [(n + 1)/4]I)w_k$. Thus $MM^t$ and $[[(n - 3)/4]J + [(n + 1)/4]I]$ agree on the orthonormal basis, so $M$ is a Hadamard tournament matrix.

**Remark 3.1.** The last part of Theorem 3.2 can be deduced at least two other ways: by noting that $M - [(n - 1)/2]I)(M^2 + M + [(n + 1)/4]I) = 0$ and using the fact that the null space of $M - [(n - 1)/2]I$ is spanned by $j$; or by applying a theorem of Ryser (see [18]) characterizing $(0,1)$ matrices with maximum determinant as design matrices.

4. A FAMILY OF TOURNAMENT MATRICES WITH DISTINCT EIGENVALUES

Recall that $U_n$ is obtained from the transitive tournament matrix $T_n$ by exchanging its $(1,n)$ and $(n,1)$ entries. Each $U_n$ is irreducible, since its directed graph has the Hamilton cycle $(1,2,\ldots, n - 1, n, 1)$. Let $p_n(\lambda) = \det(\lambda I - U_n)$.

We will use the following technique for calculating $p_n(\lambda)$ and other characteristic polynomials (see [8, p. 32], [14]). Given an $n \times n$ $(0,1)$ matrix with 0 diagonal, we denote its associated directed graph by $D$. Let $S$ be a union of vertex disjoint directed cycles in $D$; such an $S$ is called a disjoint cycle union. Let $v(S)$ be the number of vertices in $S$, and let $c(S)$ be the number of cycles in $S$. Then

$$\det(\lambda I - M) = \sum_S (-1)^{c(S)} \lambda^{n - v(S)}, \quad (4.1)$$

where the sum is taken over all disjoint cycle unions in $D$ (including the empty one).
THEOREM 4.1. For each $n \geq 3$, $p_n(\lambda) = \lambda^{n-2}(\lambda^2 + 1) - (\lambda + 1)^{n-2}$.

Proof. The directed graph associated with $U_n$ is shown in Figure 1. Each nonempty disjoint cycle union in the graph is a single cycle of the form $(n, 1, i_1, i_2, \ldots, i_k, n)$, where $2 \leq i_1 < i_2 < \cdots < i_k \leq n - 1$. Thus the nonempty disjoint cycle unions are in one-to-one correspondence with the subsets of $\{2, 3, \ldots, n - 1\}$. It follows that

$$p_n(\lambda) = \lambda^n - \sum_{k=1}^{n-2} \binom{n-2}{k} \lambda^{n-k-2} = \lambda^n + \lambda^{n-2} - (\lambda + 1)^{n-2}.\quad \blacksquare$$

THEOREM 4.2. The eigenvalues of $U_n$ all have algebraic multiplicity one. Further, when $n$ is odd, the only real eigenvalue of $U_n$ is the Perron value, while if $n$ is even, there is only one more real eigenvalue, a negative one which is strictly greater than $-\frac{1}{2}$.

Proof. Since $p_n(0) \neq 0$, we have $p_n(\lambda) = 0$ if and only if $\lambda^2 + 1 = (1 + \lambda^{-1})^{n-2}$. Set $g(\lambda) = \lambda^2 + 1$ and $h_n(\lambda) = (1 + \lambda^{-1})^{n-2}$. Since $g$ is increasing (without bound) for positive $\lambda$ while $h_n$ decreases from $\infty$ to 1 for positive $\lambda$, we see that their graphs have a unique intersection for $\lambda > 0$, which corresponds to the Perron value of $U_n$. When $\lambda < 0$, note that when $n$ is odd, $h_n(\lambda) < 1$ while $g(\lambda) > 1$, so that $p_n$ has no negative roots for odd $n$. If $n$ is even, then $h_n$ increases without bound from 1, and $g$ decreases from $\frac{5}{4}$ to 1 as $\lambda$ runs between $-\frac{1}{2}$ and 0; note that if $\lambda < -\frac{1}{2}$ then $p_n(\lambda)$ cannot be 0. Thus the graphs of $h_n$ and $g$ have a unique intersection for negative $\lambda$, and it occurs for some $\lambda$ between $-\frac{1}{2}$ and 0. Hence $p_n$ has a simple negative root [in the interval $(-\frac{1}{2}, 0)$] when $n$ is even.

To show that all of the complex roots of $p_n$ are also simple, consider what happens if $p_n$ has a complex root, say $\lambda = \mu + i\nu$ ($\nu \neq 0$) of multiplicity at
least two. Certainly \( n \geq 5 \) and \( p_n(\lambda) = p'_n(\lambda) = 0 \). This yields both \((1 + \lambda^{-1})n^{-2} = \lambda^2 + 1 \) and \((1 + \lambda^{-1})n^{-3} = -2\lambda^3/(n - 2) \). Thus \( \lambda^2 + 1 - (1 + \lambda^{-1})[-2\lambda^3/(n - 2)] \), which can be rearranged as \( \lambda^3 + (n/2)\lambda^2 + (n - 2)/2 = 0 \). Substituting \( \lambda = \mu + iv \) and taking real and imaginary parts (respectively) of the last equation, we have \( \mu^3 - 3\mu\nu^2 + (n/2)\mu^2 - \nu^2) + (n - 2)/2 = 0 \) and \( \nu(3\mu^2 + n\mu - \nu^2) = 0 \). Thus \( \nu^2 = 3\mu^2 + n\mu \), which upon substitution into the latter yields \( 8\mu^3 + 4n\mu^2 + (n^2/2)\mu = (n - 2)/2 \), or \( \mu(\mu + n/4)^2 = (n - 2)/16 \). If \( \mu < 0 \), then \( \mu < -n/3 \), because \( \nu^2 = 3\mu^2 + n\mu > 0 \), contradicting the necessity of \( \mu \geq -1/2 \) if \( \lambda \) is an eigenvalue of a tournament matrix (see [2]). Hence, writing \( 4\mu + n = \delta \), \( \mu \) must be the unique positive solution to \( \mu^2 = n - 2 \), from which it follows that \( \mu < 1/n \).

Now \( |\lambda^2 + 1|^2 = |1 - 2\mu^2 - n\mu + i2\nu\mu|^2 = 16\mu^4 + 8n\mu^3 + (n^2 - 4)\mu^2 - 2n\mu + 1 \). We have \( |\lambda^2 + 1|^2 < 2 + 16/n^4 + 4/n^2 \leq 2.1856 \) because \( \mu < 1/n \) and \( n \geq 5 \). On the other hand, \( |1 + \lambda^{-1}|^2 = |1 + \lambda|/|\mu|\delta|^2 = 1 + 2/\delta + 4/\delta^2 + n/(n/\delta)^2 > 1 + (n/\delta)^2 \), the last because \( \mu\delta^2 = n - 2 \) and \( \delta > n \). Thus \( |1 + \lambda^{-1}|^2 > 1 + (1 + 4\mu/n)^2 > 1 + (1 + 4/n)^2 = 1.743 \ldots \), since \( \mu^{-1} \geq n \geq 5 \). As a result, \( (1 + \lambda^{-1})n^{-2} > (1.74)^2 > 2.1856 > |\lambda^2 + 1|^2 \), and hence \( \lambda \) cannot be a solution to \( (1 + \lambda^{-1})n^{-2} = \lambda^2 + 1 \). Consequently \( p_n \) has no nonreal roots of multiplicity two or more.

**Theorem 4.3.** The right eigenvectors of \( U_n \) corresponding to the eigenvalue \( \lambda \) are scalar multiples of \( y = (\alpha^2, \alpha^{n-3}, \alpha^{n-4}, \ldots, \alpha, 1, \lambda)^t \), where \( \alpha = 1 + \lambda^{-1} \).

**Proof.** Let \( z = U_n y \); we will show that \( z = \lambda y \). Let the \( i \)th components of \( z \) and \( y \) be \( z_i \) and \( y_i \) respectively, \( 1 \leq i \leq n \). Multiplying \( y \) by \( U_n \), we have \( z_1 = \sum_{j=0}^{n-2} \alpha^j, z_{n-1} = \lambda, z_n = \lambda^2 \), and \( z_r = \lambda + \sum_{j=0}^{n-2} (-\alpha^j) \) for \( 2 \leq r \leq n - 2 \). Evidently \( z_{n-1} = \lambda y_{n-1} \) and \( z_n = \lambda y_n \). For \( 2 < r < n - 2 \), we have \( z_r = \lambda + (1 - \alpha^{-n-1-r})(1 - \alpha^{-1-r}) = \lambda \alpha^{n-1-r} = \lambda y_r \). Similarly, \( z_1 = (1 - \alpha^{-n-2})/(1 - \alpha) = \lambda(1 + \lambda^{-1})^{-n-2} - \lambda = \lambda y_1 \), since \( \lambda \) is a root of the characteristic polynomial of \( U_n \).

A similar argument shows that a left eigenvector of \( U_n \) corresponding to \( \lambda \) is a scalar multiple of \( w = (\lambda, 1, \alpha, \alpha^2, \ldots, \alpha^{n-3}, \lambda^2) \). In particular when \( n \) is even, the right and left eigenvectors, \( y \) and \( w \) respectively, which are associated with the negative root of \( p_n(\lambda) \) have the sign pattern \( \text{sgn}(y^t) = (+, -, +, -, \ldots, +, -) = -\text{sgn}(w) \). Moreover, Theorem 4.3 also implies that no entry in any eigenvector of \( U_n \) is 0 (since 0 and -1 are never eigenvalues of \( U_n \)). See [13] for a discussion of zero entries in eigenvectors.

Tournaments and directed graphs whose adjacency matrices have distinct eigenvalues have been discussed in the graph theory literature. Specifically, Cameron [7] points out that the automorphism groups of such directed
graphs must be abelian. As the referee for this work has noted, the fact that
the automorphism group of the tournament corresponding to $U_n$ is abelian
also follows from its degree sequence.

5. MINIMUM PERRON VALUE AND THE MULTIPLICITY OF ZERO

Let $\pi_n$ denote the minimum possible Perron value over the class of
$n \times n$ irreducible tournament matrices. Let $\alpha_n$ be the maximum possible
algebraic multiplicity of zero over the same class (we remark that $\alpha_n = 0$ if
$3 \leq n \leq 5$). Our first result relates these two quantities.

**Lemma 5.1.** Let $M$ be an irreducible tournament matrix of order $n \geq 3$
with Perron value $\rho$. Let $m_0$ be the algebraic multiplicity of 0 as an
eigenvalue of $M$. Then $m_0 \leq n - 1 - 2\pi$, and in particular $\alpha_n \leq n - 1 - 2\pi_n$.

*Proof.* Let the spectrum of $M$ be $\{\rho, \lambda_2, \lambda_3, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_n\}$, where
$\text{Re} \lambda_i \neq 0$ if and only if $i \leq r$. Since the trace of $M$ is zero, we have
$\rho = -\sum_{i=2}^{r} \text{Re} \lambda_i$. But $\text{Re} \lambda_i \geq -\frac{1}{2}$ for $2 \leq i \leq n$, so $r \geq 2\rho + 1$. Thus we have
$m_0 \leq n - r \leq n - 1 - 2\rho$, from which it follows that $\alpha_n \leq n - 1 - 2\pi_n$. \qed

The lemma above suggests that in order to find an upper bound on $\alpha_n$,
we should try to estimate $\pi_n$. In [3], Brauer and Gentry showed that $\pi_n > \frac{3}{\sqrt{2}}$
for $n \geq 3$. We will improve that bound and show that the sequence of $\pi_n$'s is
bounded above in our next theorem.

**Theorem 5.1.** For all $n \geq 8$, $2 < \pi_n < 2.5$.

*Proof.* We consider the family of tournament matrices defined by $S_n =
[s_{ij}]$, $i, j = 1, 2, \ldots, n$, with $s_{ij} = 1$ if and only if $j > i + 1$ or $j = i - 1$. Note
that $S_n$ is obtained from the transitive tournament $T_n$ by exchanging the
entries in the first sub- and superdiagonals. Thus

$$S_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad S_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad S_4 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$
The transposes of these matrices were studied by Katzenberger and Shader [12]. Note that each $S_n$ of order $n \geq 3$ is irreducible, since its digraph has the Hamilton cycle $(n, n-1, \ldots, 2, 1, n)$. The directed graph $D_n$ associated with $S_n$ is pictured in Figure 2.

Let $L_1(\lambda) = \lambda$ and $L_n(\lambda) = \det(\lambda I - S_n)$ for $n \geq 2$; we claim that for $n \geq 4$

$$L_n(\lambda) = (\lambda + 1)L_{n-1}(\lambda) - \lambda L_{n-2}(\lambda) - L_{n-3}(\lambda).$$

To show the claim, first partition the disjoint cycle unions in $D_n$ into sets $T_1$, $T_2$, and $T_3$, where a disjoint cycle union $S$ in $D_n$ is in $T_1$ if the arc $(n, n-1)$ is not in $S$, in $T_2$ if the path $(n-2, n, n-1)$ is in $S$, and in $T_3$ if the arc $(n-2, n)$ is in $S$ but the arc $(n-2, n)$ is not. We have

$$l_n(\lambda) = \sum_{S \in T_1} (-1)^{c(S)}\lambda^{n-\nu(S)} + \sum_{S \in T_2} (-1)^{c(S)}\lambda^{n-\nu(S)} + \sum_{S \in T_3} (-1)^{c(S)}\lambda^{n-\nu(S)}.$$

Since each $S$ in $T_1$ is in $D_{n-1}$, the first sum in (5.2) is $\lambda l_{n-1}(\lambda)$. If $S$ is in $T_2$, then it consists of the 3-cycle $(n, n-1, n-2, n)$ along with a disjoint cycle union in $D_{n-3}$, so that the second sum in (5.2) is $-l_{n-3}(\lambda)$. Finally,
the disjoint cycle unions $S$ in $T_3$ are in one-to-one correspondence with the disjoint cycle unions in $D_{n-1}$ which involve the arc $(n-1, n-2)$ (i.e. those not in $T_1$ in $D_{n-1}$). It follows that the third sum in (5.2) is $l_{n-1}(\lambda) - \lambda l_{n-2}(\lambda)$, which yields (5.1). Note that setting $\lambda = 0$ in Eq. (5.1) yields the recurrence relation for the determinants of the $S_n$'s that was found by Katzenberger and Shader in [12].

We will now show by induction that if $\lambda \geq 2.5$ and $n \geq 1$, then $l_n(\lambda) \geq 2l_{n-1}(\lambda) > 0$; this can be verified directly if $n < 3$. If $\lambda \geq 2.5$ and $n > 4$, then by the induction hypothesis we have

$$l_n(\lambda) = (\lambda + 1)l_{n-1}(\lambda) - \lambda l_{n-2}(\lambda) - l_{n-3}(\lambda)$$

$$\geq \left( \frac{\lambda}{2} + \frac{3}{4} \right) l_{n-1}(\lambda) \geq 2l_{n-1}(\lambda) > 0.$$  

It follows that the Perron value of $S_n$, $\gamma_n$ say, is less than 2.5 for any $n \geq 3$, so that $\pi_n < 2.5$ for $n \geq 3$. A result of Moser and Harary (see [16, Theorem 3, p. 6]) implies that when $n > 3$, every irreducible tournament matrix of order $n$ contains an irreducible principal submatrix of order $n - 1$ (also a tournament matrix). It follows from a theorem of Wielandt (see [11, Theorem 8.4.5, p. 509]) that irreducible principal submatrices of an irreducible matrix $A$ have Perron values at most that of $A$. Thus we see that $\pi_n \leq \gamma_{n+1}$ for $n \geq 3$ (indeed, a similar argument shows that $\gamma_n \leq \gamma_{n+1}$ for $n \geq 3$). A computer search of all irreducible tournament matrices of order 8 (using a disk listing all tournament matrices of orders $n < 8$ kindly supplied by Professor R. Read) showed that $\gamma_8 = \pi_8 = 2.0606\ldots$ and hence $2 < \pi_n < 2.5$ for all $n \geq 8$.

Both the $\pi_n$'s and the $\gamma_n$'s are nondecreasing and bounded above, so it is natural to wonder about their respective limits. Led by an observation of the referee on the characteristic equation of the recurrence for $l_n$, we have been able to show that

$$\gamma_n \rightarrow -1 + \frac{\sqrt{13 + 16\sqrt{2}}}{2} = 2.4844353\ldots$$

as $n \rightarrow \infty$.

Bruaidi and Li [6] conjecture that $\gamma_n = \pi_n$ for any $n \geq 3$; numerical results using Professor Read's disk show that the latter statement holds when $3 \leq n \leq 8$.

Applying the lower bound on $\pi_n$ and Lemma 5.1, we have the following result.
**Corollary 5.1.1.** For \( n \geq 8 \), \( \alpha_n \leq n - 6 \).

We now examine a class of irreducible tournament matrices that yield a lower bound on \( \alpha_n \).

For \( p, q \geq 1 \), denote the \( p \times q \) matrix of ones by \( J_{p,q} \) and of zeros by \( O_{p,q} \). We define

\[
M_{k,l} = \begin{bmatrix}
U_k & J_{k-1,l} & 0 \\
0 & 1 & 1 \\
0 & 1 & 1 & \ldots & 1 \\
O_{l-1,k} & & & & U_l
\end{bmatrix}
\]

for all \( k, l \geq 3 \).

Evidently, each \( M_{k,l} \) is a tournament matrix. In the terminology of Brualdi and Li [5], \( M_{k,l} \) is the \textit{join} of \( U_k \), \( U_2 \), and \( U_l \). A result of Katzenberger and Shader [12] ensures that \( M_{k,l} \) is singular for \( l, k \geq 3 \); the theorem below will find the algebraic multiplicity of 0 as an eigenvalue of \( M_{k,l} \).

**Theorem 5.2.** For each \( l, k \geq 3 \), \( M_{k,l} \) is an irreducible tournament matrix and

\[
\det(\lambda I - M_{k,l}) = \lambda^{k+l-4}(1 + \lambda^2)^5 - (1 + \lambda + \lambda^2) \\
\times \left[ \lambda^{k-2}(1 + \lambda)^{l-2} + \lambda^{l-2}(1 + \lambda)^{k-2} \right].
\]

**Proof.** \( M_{k,l} \) is irreducible, since its associated directed graph \( D \) has the Hamilton cycle \((k + l, k + 1, k, 1, 2, \ldots, k - 1, k + 2, k + 3, \ldots, k + l - 1, k + l)\). The subgraph of \( D \) induced by the vertices 1, \( k \), \( k + 1 \), and \( k + l \) is shown in Figure 3.

We partition the disjoint cycle unions in \( D \) into sets \( T_1 \), \( T_2 \), \( T_3 \), and \( T_4 \), where a disjoint cycle union \( S \) is in \( T_1 \) if the arc \((k + 1, k)\) is not in \( S \), in \( T_2 \) if

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**Fig. 3.**
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the arcs \((k+1,k)\) and \((k,1)\) are in \(S\) but the arc \((k+l,k+1)\) is not, in \(T_3\), if the arcs \((k+1,k)\) and \((k+l,k+1)\) are in \(S\) but the arc \((k,1)\) is not, and in \(T_4\) if the arcs \((k+1,k)\), \((k+l,k+1)\) and \((k,1)\) are in \(S\). Thus we have

\[
\det(\lambda I - M_{k,l}) = \sum_{S \in T_1} (-1)^{c(S)} \lambda^{n-\nu(S)} + \sum_{S \in T_2} (-1)^{c(S)} \lambda^{n-\nu(S)}
\]

\[
+ \sum_{S \in T_3} (-1)^{c(S)} \lambda^{n-\nu(S)} + \sum_{S \in T_4} (-1)^{c(S)} \lambda^{n-\nu(S)}. \quad (5.3)
\]

The disjoint cycle unions in \(T_1\) are unions of a disjoint cycle union in the graph of \(U_k\) and disjoint cycle union in the graph of \(U_l\); thus the first sum in (5.3) is \(p_k(\lambda)p_l(\lambda)\). If \(S\) is in \(T_2\), then it is \((k+1,k,1,k+1)\) or a single cycle of the form \((k+1,k,1,i_1,\ldots,i_q,k+1)\), where \(2 < i_1 < \cdots < i_q < k-1\). It follows that the second sum in (5.3) is

\[
- \sum_{q=0}^{k-2} \binom{k-2}{q} \lambda^{k+1-q-3} = -\lambda^{k-1}(1+\lambda)^{k-2}.
\]

A similar argument shows that the third sum in (5.3) is \(-\lambda^{k-1}(1+\lambda)^{l-2}\). Finally, if \(S\) is in \(T_4\), then it is \((k+1,k+1,k,1)\), or a single cycle of the form

\[
(k+l,k+1,k,1,i_1,\ldots,i_q,j_1,\ldots,j_p,k+1),
\]

where \(2 < i_1 < \cdots < i_q < k-1\) and \(k+2 < j_1 < \cdots < j_p < k+l-1\). It follows that the last sum in (5.3) is \(-(1+\lambda)^{k+l-4}\). Hence we find that

\[
\det(\lambda I - M_{k,l}) = p_k(\lambda)p_l(\lambda) - \lambda^{l-1}(1+\lambda)^{k-2} - \lambda^{k-1}(1+\lambda)^{l-2} - (1+\lambda)^{k+l-4},
\]

and applying the formula for \(p_n(\lambda)\) in Theorem 4.1 yields the result.

\[\blacksquare\]

**Remark 5.1.** It is clear from its definition that \(M_{k,l}\) is obtained from the reducible tournament matrix

\[
\bar{M} = \begin{bmatrix}
U_k & I_{k,l} \\
O_{l,k} & U_l
\end{bmatrix}
\]

by exchanging the \((k,k+1)\) and \((k+1,k)\) entries. Note that \(\det(\lambda I - \bar{M}) = \det(\lambda I - U_k)\det(\lambda I - U_l)\). Thus the effect of reversing the \((k,k+1)\) and
(k + 1, k) entries in $\lambda I - \tilde{M}$ (to obtain $\lambda I - M_{k,k}$) is to introduce the term 

$$-\left[\lambda^{k-1}(1 + \lambda)^{l-2} + \lambda^{l-1}(1 + \lambda)^{k-2} + (1 + \lambda)^{k+l-4}\right]$$

into the determinant.

**Theorem 5.3.** For every $n \geq 6$ and every $1 \leq m \leq \lfloor n/2 \rfloor - 2$ there is an irreducible tournament matrix of order $n$ having 0 as an eigenvalue with algebraic multiplicity $m$.

**Proof.** From our formula in Theorem 5.2, we see that $M_{k,l}$ has 0 as an eigenvalue of algebraic multiplicity $\min(k,l) - 2$. Thus, for each $n \geq 6$ and each $1 \leq m \leq \lfloor n/2 \rfloor - 2$, $M_{m,\lfloor n/2 \rfloor - 2}$ has 0 as an eigenvalue of algebraic multiplicity $m$. \hfill $\blacksquare$

**Corollary 5.3.1.** For $n \geq 8$, $\lfloor n/2 \rfloor - 2 \leq \alpha_n \leq n - 6$.

Using Professor Read’s disk, we found that $\alpha_6 = \alpha_7 = 1$ and $\alpha_8 = 2$. Further, the $9 \times 9$ tournament matrix below in Example 5.1 is irreducible and has 0 as an eigenvalue of algebraic multiplicity 3. Thus, $\alpha_9 = 3$ from Corollary 5.3.1.

**Example 5.1.** Let

$$A = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}.$$ 

Its directed graph has the Hamilton cycle (9,7,6,4,1,2,3,5,8,9), so $A$ is irreducible. Its characteristic polynomial is $\lambda^9 - 7\lambda^6 - 8\lambda^5 - 9\lambda^4 - 3\lambda^3$. Thus we see that when $n = 9$, the lower bound on $\alpha_n$ (given in Corollary 5.3.1) is too small, while the upper bound is sharp.

**Remark 5.2.** Each of the matrices in Theorems 4.1, 5.1, and 5.2 and in Example 5.1 has the row sum vector $(n-2, n-2, n-3, n-4 \ldots 3, 2, 1, 1, 1)$. Because of a characterization of tournament matrices with this row sum vector given by Brualdi and Li (see [5]), the disjoint cycle union technique for finding characteristic polynomials works well for such matrices.
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