Spectrally arbitrary star sign patterns

G. MacGillivray *, R.M. Tifenbach, P. van den Driessche

Department of Mathematics and Statistics, University of Victoria, P.O. Box 3045, Victoria BC, Canada V8W 3P4

Received 30 August 2004; accepted 8 November 2004

Available online 20 January 2005

Submitted by R.A. Brualdi

Abstract

An $n \times n$ sign pattern $S_n$ is spectrally arbitrary if, for any given real monic polynomial $g(x)$ of degree $n$, there is a real matrix having sign pattern $S_n$ and characteristic polynomial $g(x)$. All $n \times n$ star sign patterns that are spectrally arbitrary, and all minimal such patterns, are characterized. This subsequently leads to an explicit characterization of all $n \times n$ star sign patterns that are potentially nilpotent. It is shown that any super-pattern of a spectrally arbitrary star sign pattern is also spectrally arbitrary.

AMS classification: 05C50; 15A18

Keywords: Inertially arbitrary; Potentially nilpotent; Potentially stable; Spectrally arbitrary; Star sign pattern

* Research supported in part by NSERC through an USRA for R.M. Tifenbach and Discovery Grants for G. MacGillivray and P. van den Driessche.

* Corresponding author.

E-mail addresses: gmacgill@math.uvic.ca (G. MacGillivray), ryant@math.uvic.ca (R.M. Tifenbach), pvdd@math.uvic.ca (P. van den Driessche).

0024-3795/$ - see front matter © 2004 Elsevier Inc. All rights reserved.
1. Introduction

Our goal is to characterize spectrally arbitrary star sign patterns. To explain this goal, we start with some definitions and then put our problem in the context of other results on sign patterns.

An $n \times n$ array whose entries are chosen from the set $\{+, -, 0\}$ is called a sign pattern. Let $S_n = [s_{ij}]$ be an $n \times n$ sign pattern $(n \geq 2)$. A real $n \times n$ matrix $M_n = [m_{ij}]$ has sign pattern $S_n$ if $\text{sgn}(m_{ij}) = s_{ij}$ for all $i$ and $j$. If, for any given real monic polynomial $g(x)$ of degree $n$, there is a real matrix having sign pattern $S_n$ and characteristic polynomial $g(x)$, then $S_n$ is a spectrally arbitrary sign pattern (SAP).

Thus, $S_n$ is a SAP if, given any self-conjugate spectrum, there is a real matrix having sign pattern $S_n$ and that spectrum. Spectrally arbitrary sign patterns are normally identified up to equivalence, since the property of being a SAP is clearly preserved under negation, transposition, signature similarity and permutation similarity. If there is a real matrix having sign pattern $S_n$ and characteristic polynomial $g(x) = x^n$, then $S_n$ is potentially nilpotent; in particular each SAP must be potentially nilpotent.

The inertia of an $n \times n$ matrix $M_n$ is the ordered triple $I(M_n) = (i_+, (M_n), i_-(M_n), i_0(M_n))$ where $i_+ (M_n)$, $i_- (M_n)$ and $i_0(M_n)$ are the number of eigenvalues of $M_n$ with positive, negative and zero real parts, respectively. The sign pattern $S_n$ is an inertially arbitrary sign pattern (IAP) if, for any ordered triple of non-negative integers $(i_1, i_2, i_3)$ such that $i_1 + i_2 + i_3 = n$, there is a real matrix $M_n$ having sign pattern $S_n$ such that $I(M_n) = (i_1, i_2, i_3)$. The sign pattern $S_n$ is potentially stable if there is a real matrix $M_n$ having sign pattern $S_n$ and inertia $I(M_n) = (0, n, 0)$. From the definitions, every SAP is an IAP and every IAP is potentially stable.

A super-pattern of $S_n$ is a sign pattern $R_n = [r_{ij}]$ such that if $s_{ij} \neq 0$, then $r_{ij} = s_{ij}$. If $R_n$ is a super-pattern of $S_n$, then $S_n$ is a sub-pattern of $R_n$. If sign pattern $S_n$ is an SAP (IAP) but none of its proper sub-patterns is a SAP (IAP), then $S_n$ is a minimal SAP (IAP).

A sign pattern $S_n = [s_{ij}]$ is combinatorially symmetric if $s_{ij} \neq 0$ whenever $s_{ji} \neq 0$. The graph $G(S_n)$ of a combinatorially symmetric sign pattern has vertices 1, 2, $\ldots$, $n$ and an edge joining vertices $i$ and $j$ if and only if $s_{ij} \neq 0$. Note that loops are allowed. A star is the graph with vertices 1, 2, $\ldots$, $n$ and an edge joining a fixed centre vertex $i$ and each leaf vertex $j$ for all $j \neq i$ (and no other edges). In what follows, it is assumed that 1 is the centre vertex. A combinatorially symmetric sign pattern $S_n$ is a star sign pattern if the graph obtained from $G(S_n)$ by deleting all loops is a star, and more generally, is a tree sign pattern if this graph is a tree.

Spectrally arbitrary tree sign patterns, especially those whose graph (excluding loops) is a path, are considered in [1]. A method, based on the implicit function theorem, for proving that a pattern (and all super-patterns) is a SAP is developed there. A class of (full) spectrally arbitrary patterns is constructed in [2] by using a Soules matrix. The implicit function theorem method is used in [3] to show that some Hessenberg sign patterns are minimal SAPs, the first such families for all orders to be presented. Other spectrally arbitrary sign pattern classes are constructed in [4].
also by using the implicit function theorem method. All potentially stable star sign patterns are characterized in [5]; we make use of this characterization in Section 5. The inertias of matrices having a symmetric star sign pattern are characterized in [6]. Potentially nilpotent star sign patterns are considered in [7], in which explicit characterizations are given for patterns of orders two and three, and a recursive characterization for patterns of general order \( n \) is proved. Further aspects of these sign pattern problems can be found in references of the papers cited above.

We begin with some preliminary results (Section 2) and use these to prove that two families of star sign patterns, \( Z_{np} \) and \( Y_n \), are spectrally arbitrary (Section 3). We then use the implicit function theorem method to show that any super-pattern of these is also a SAP (Section 4). We show that \( Z_{np} \) and \( Y_n \) are the only minimal inertially arbitrary star sign patterns (Section 5). We characterize all star sign patterns that are potentially nilpotent (Section 6), and conclude (Section 7) with a summary of our results for star sign patterns.

2. Preliminary results

Throughout this paper, we use the following conventions. The lower-case roman letters \( a, b, c \) and \( d \) (with or without subscripts) denote positive real numbers. In all matrices and arrays, entries that are not specified are equal to zero.

Let \( M_n \) be the real \( n \times n \) matrix

\[
M_n = \begin{bmatrix}
\alpha_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_n \\
\beta_2 & \alpha_2 & 0 & \cdots & 0 \\
\beta_3 & 0 & \alpha_3 & \cdots & 0 \\
& \ddots & \ddots & \ddots & \ddots \\
\beta_n & 0 & \cdots & 0 & \alpha_n
\end{bmatrix}
\]

with characteristic polynomial \( \det(xI_n - M_n) \), denoted by \( f_n(x) \). Expanding this determinant about the first row gives the following expression for \( f_n(x) \).

**Lemma 2.1.** The characteristic polynomial of \( M_n \) (as above) is

\[
f_n(x) = \prod_{j=1}^{n} (x - \alpha_j) - \sum_{i=2}^{n} \left( \beta_i \gamma_i \prod_{j \neq i \atop j \in \{2, \ldots, n\}} (x - \alpha_j) \right)
\]

Note that if \( \beta_i = 0 \) or \( \gamma_i = 0 \) for some \( i \), then \( \alpha_i \) is a zero of \( f_n(x) \), and hence an eigenvalue of \( M_n \). Thus, such a matrix has an eigenvalue of fixed sign, so its sign pattern cannot be an IAP or a SAP. From Lemma 2.1, the off-diagonal entries \( \beta_i \) and \( \gamma_i \) enter into the characteristic polynomial of \( M_n \) only as a product \( \beta_i \gamma_i \). In a spectrally arbitrary star sign pattern the entries \((1, i)\) and \((i, 1)\) for \( 2 \leq i \leq n \) are non-zero (the sign pattern is irreducible). It is therefore sufficient to consider only
matrices with \( y_i = 1 \) for \( 2 \leq i \leq n \) and for star sign patterns it is sufficient to take entries \((1, j)\) for \( 2 \leq j \leq n \) as +.

**Lemma 2.2.** Let \( n \geq 2 \). If \( S_n = [s_{ij}] \) is an inertially arbitrary star sign pattern, then \( s_{ii} \neq 0 \) for any leaf \( i \).

**Proof.** Let \( n \geq 2 \) and let \( S_n \) be an inertially arbitrary star sign pattern. Let the real matrix

\[
M_n = \begin{bmatrix}
\alpha_1 & 1 & 1 & \cdots & 1 \\
\beta_2 & \alpha_2 & & & \\
\beta_3 & & \alpha_3 & & \\
& \ddots & & \ddots & \\
\beta_n & & & \cdots & \alpha_n
\end{bmatrix}
\]

have sign pattern \( S_n \). Then,

\[
\det(M_n) = \prod_{j=1}^{n} \alpha_j - \sum_{i=2}^{n} \left( \beta_i \prod_{2 \leq j, \alpha_j \neq i} \alpha_j \right)
\]

By definition of a star sign pattern, \( \beta_i \neq 0 \) for all \( i \). Suppose that \( \alpha_i = 0 \) for exactly one \( i \) with \( 2 \leq i \leq n \). Then \( \det(M_n) \neq 0 \). So \( M_n \) has only non-zero eigenvalues and, thus, the eigenvalues that have zero real parts must be conjugate pairs of purely imaginary numbers. Therefore \( i_0(M_n) \) is even for any real matrix \( M_n \) with sign pattern \( S_n \), implying that \( S_n \) is not inertially arbitrary, and giving a contradiction.

Now suppose that \( \alpha_i = 0 \) for more than one \( i \) with \( 2 \leq i \leq n \). Then \( \det(M_n) = 0 \). But this implies that \( i_0(M_n) \geq 1 \) for all \( M_n \) with sign pattern \( S_n \), again giving a contradiction. Therefore, \( s_{ii} \neq 0 \) for all \( 2 \leq i \leq n \). \( \square \)

**Corollary 2.3.** Let \( n \geq 2 \). If \( S_n = [s_{ij}] \) is a spectrally arbitrary star sign pattern, then \( s_{ii} \neq 0 \) for any leaf \( i \).

**Lemma 2.4.** Let \( n \geq 2 \) and \( M_n \) (as in Lemma 2.2) be a real matrix having a star sign pattern. If \( M_n \) is nilpotent, then the non-zero entries among \( \alpha_2, \alpha_3, \ldots, \alpha_n \) are distinct.

**Proof.** By Lemma 2.1, the characteristic polynomial of \( M_n \) is

\[
f_n(x) = \prod_{j=1}^{n} (x - \alpha_j) - \sum_{i=2}^{n} \left( \beta_i \prod_{2 \leq j, \alpha_j \neq i} (x - \alpha_j) \right)
\]

If \( \alpha_i = \alpha_j \) for some \( i, j \geq 2 \) with \( i \neq j \), then \( \alpha_i \) is a zero of \( f_n(x) = x^n \), and consequently \( \alpha_i = 0 \). Thus, if \( \alpha_i \neq 0 \), then \( \alpha_i \neq \alpha_j \) for all \( i, j \geq 2 \) with \( i \neq j \). \( \square \)
The following two results are used in our constructions of spectrally arbitrary star sign patterns.

**Lemma 2.5.** Let \( g(x) = x^n + \sum_{i=0}^{n-1} \mu_i x^i \) be a real monic polynomial of degree \( n \geq 1 \). Let \( t = \max(\sum_{i=0}^{n-1} |\mu_i|, 1) \). Then \( t \) is greater than or equal to the absolute value of any zero of \( g(x) \). Furthermore, if \( |\gamma| > t \) and \( \gamma \) is real, then \( \text{sgn}(g(\gamma)) = \text{sgn}(\gamma^n) \).

**Proof.** Suppose \( |\gamma| > t = \max(\sum_{i=0}^{n-1} |\mu_i|, 1) \). Then

\[
|g(\gamma) - \gamma^n| \leq \sum_{i=0}^{n-1} |\mu_i| |\gamma|^i \leq \sum_{i=0}^{n-1} |\mu_i| (|\gamma| - 1) \sum_{i=0}^{n-1} |\mu_i| < |\gamma|^n
\]

Therefore \( g(\gamma) \neq 0 \). So, the absolute value of any zero of \( g(x) \) must be at most \( t \). Moreover, since \( g(x) \) is monic, \( g(x) \rightarrow \infty \) as \( x \rightarrow \infty \) and \( g(x) \rightarrow \text{sgn}(-1)^p \infty \) as \( x \rightarrow -\infty \), thus \( \text{sgn}(g(\gamma)) = \text{sgn}(\gamma^n) \) for \( |\gamma| > t \). □

**Lemma 2.6.** Suppose \( a_p < a_{p-1} < \cdots < a_1 \) and \( b_1 < b_2 < \cdots < b_q \) are positive real numbers (where possibly \( p = 0 \) or \( q = 0 \)). If \( p \neq 0 \), then for each fixed \( i \) with \( 1 \leq i \leq p \),

\[
\text{sgn} \left( \prod_{1 \leq j < p \atop j \neq i} (-a_i + a_j) \prod_{k=1}^{q} (-a_i - b_k) \right) = \text{sgn}((-1)^{p+q-i})
\]

Further, if \( q \neq 0 \), then for each fixed \( i \) with \( 1 \leq i \leq q \),

\[
\text{sgn} \left( \prod_{j=1}^{p} (b_i + a_j) \prod_{1 \leq k < q \atop k \neq i} (b_i - b_k) \right) = \text{sgn}((-1)^{q-i})
\]

**Proof**

\[
\prod_{1 \leq j < p \atop j \neq i} (-a_i + a_j) \prod_{k=1}^{q} (-a_i - b_k)
\]

\[
= (-1)^{(p-i)+q} \prod_{j=1}^{i-1} (a_j - a_i) \prod_{j=i+1}^{p} (a_i - a_j) \prod_{k=1}^{q} (a_i + b_k)
\]

Since each term in the product on the right is positive, the product on the left has the stated sign. Similarly,

\[
\prod_{j=1}^{p} (b_i + a_j) \prod_{1 \leq k < q \atop k \neq i} (b_i - b_k)
\]
\[ = (-1)^{q-i} \prod_{j=1}^{p} (b_j + a_j) \prod_{k=1}^{i-1} (b_k - b_k) \prod_{k=i+1}^{q} (b_k - b_i) \]
giving the second statement. \( \square \)

3. Spectrally arbitrary star sign patterns

As in the note after Lemma 2.1, the first row of a star sign pattern can be assumed to be \([s_{11} + + \cdots +] \).

Let \( n \geq 3 \) and \( 1 \leq p \leq n - 2 \). Let \( Z_{np} = [z_{ij}] \) be the \( n \times n \) star sign pattern with:

\[
\begin{align*}
z_{11} &= 0 \\
z_{ii} &= \begin{cases} - & \text{if } 2 \leq i \leq p + 1 \\ + & \text{if } p + 2 \leq i \leq n \end{cases} \\
z_{i1} &= \begin{cases} \sgn((-1)^{i-1}) & \text{if } 2 \leq i \leq p + 1 \\ \sgn((-1)^{n-i+1}) & \text{if } p + 2 \leq i \leq n \end{cases}
\end{align*}
\]

Thus,

\[
Z_{np} = \begin{bmatrix}
0 & + & + & \cdots & + & \cdots & \cdots & + \\
- & - & + & \cdots & \cdots & \cdots & \cdots & - \\
+ & + & + & \cdots & \cdots & \cdots & + & + \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\sgn((-1)^p) & \sgn((-1)^{n-p-1}) & + & + & + & + & + & + \\
\sgn((-1)^{n-p-2}) & + & + & + & + & + & + & + \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
- & + & + & \cdots & + & \cdots & \cdots & +
\end{bmatrix}
\]

Note that \( Z_{np} \) has \( p \geq 1 \) negative entries and \( q = n - p - 1 \geq 1 \) positive entries among the entries \( z_{ii} \).

**Theorem 3.1.** The star sign pattern \( Z_{np} \) (\( n \geq 3 \) and \( 1 \leq p \leq n - 2 \)) is spectrally arbitrary.

**Proof.** Let \( n \geq 3 \) and \( 1 \leq p \leq n - 2 \) be positive integers. Define \( q = n - p - 1 \) and note that \( 1 \leq q \leq n - 2 \) and \( p + q + 1 = n \). Let \( g(x) = x^n + \sum_{i=0}^{n-1} \mu_i x^i \) be any real monic polynomial of degree \( n \). Find positive \( t \) greater than or equal to the absolute value of any zero of \( g(x) \); for example, as in Lemma 2.5, pick \( t = \max\{\sum_{i=0}^{n-1} |\mu_i|, 1\} \). Pick positive numbers \( a_1, a_2, \ldots, a_p \) and \( b_1, b_2, \ldots, b_q \) such that
\[ t < a_p < a_{p-1} < \cdots < a_1 \quad t < b_1 < b_2 < \cdots < b_q \]

and
\[ \sum_{j=1}^{p} a_j - \sum_{k=1}^{q} b_k = \mu_{n-1} \]

This is always possible: first pick the numbers so that they satisfy the inequalities, then increase either \( a_1 \) or \( b_q \) until equality in the third condition is satisfied. For each \( i \) with \( 1 \leq i \leq p \), let
\[ c_i = \frac{(-1)^{i+1} g(-a_i)}{\prod_{j \neq i} (-a_i + a_j) \prod_{k=1}^{q} (-a_i - b_k)} \]

and for each \( i \) with \( 1 \leq i \leq q \), let
\[ d_i = \frac{(-1)^{q+i} g(b_i)}{\prod_{j=1}^{p} (b_i + a_j) \prod_{k \neq i} (b_i - b_k)} \]

By Lemmas 2.5 and 2.6, \( \text{sgn}(c_i) = + \) and \( \text{sgn}(d_i) = + \).

Consider the following matrix \( M_n \) having sign pattern \( Z_{np} \):

\[
M_n = \begin{bmatrix}
0 & 1 & \ldots & \ldots & \ldots & \ldots & \ldots & 1 \\
-c_1 & -a_1 & & & & & & \\
-c_2 & -a_2 & & & & & & \\
\vdots & & & & & & & \\
(-1)^p c_p & & & & & & & \\
(-1)^q d_1 & & & & & & & b_1 \\
(-1)^{q-1} d_2 & & & & & & & b_2 \\
\vdots & & & & & & & \\
-d_q & & & & & & & b_q \\
\end{bmatrix}
\]

Note that \( a_i, b_i, c_i \) and \( d_i \) are positive for all \( i \), and for the entries in \( M_n \), \( c_i \) is multiplied by \( (-1)^i \) and \( d_i \) is multiplied by \( (-1)^{q-i+1} \). By Lemma 2.1, the characteristic polynomial of \( M_n \) is

\[
f_n(x) = x^p \prod_{j=1}^{p} (x + a_j) \prod_{k=1}^{q} (x - b_k)
+ \sum_{i=1}^{p} \left( (-1)^{i+1} c_i \prod_{\substack{j \neq i \atop j \neq i}}^{1} (x + a_j) \prod_{k=1}^{q} (x - b_k) \right)
+ \sum_{i=1}^{q} \left( (-1)^{q-i} d_i \prod_{j=1}^{p} (x + a_j) \prod_{\substack{k \neq i \atop k \neq i}}^{1} (x - b_k) \right)
\]
This polynomial is monic of degree $n$, and the coefficient of $x^{n-1}$ is

$$\sum_{j=1}^{p} a_j - \sum_{k=1}^{q} b_k = \mu_{n-1}$$

which is the coefficient of $x^{n-1}$ in $g(x)$. For each $i$ with $1 \leq i \leq p$

$$f_n(-a_i) = (-1)^{j+1} c_i \prod_{1 \leq j \leq p, j \neq i} (-a_i + a_j) \prod_{k=1}^{q} (-a_i - b_k) = g(-a_i)$$

and for each $i$ with $1 \leq i \leq q$

$$f_n(b_i) = (-1)^{j-i} d_i \prod_{1 \leq j \leq q, j \neq i} (b_i + a_j) \prod_{k=1}^{p} (b_i - b_k) = g(b_i)$$

Since $f_n(x)$ and $g(x)$ are both monic and have the same coefficients of $x^{n-1}$, the polynomial $f_n(x) - g(x)$ has degree at most $n - 2$. Furthermore, since $-a_1, -a_2, \ldots, -a_p, b_1, b_2, \ldots, b_q$ are distinct, $f_n(x) - g(x)$ has $n - 1$ distinct zeroes. Therefore $f_n(x) - g(x) = 0$. The characteristic polynomial of $M_n$ is $g(x)$ and $M_n$ has sign pattern $Z_{np}$. Thus, $Z_{np}$ is a spectrally arbitrary star sign pattern. □

Let $n \geq 2$ and $Y_n = [y_{ij}]$ be the $n \times n$ star sign pattern with:

$$y_{11} = +$$

$$y_{ii} = - \quad \text{if } i \neq 1$$

$$y_{11} = \text{sgn}(-1)^{i-1} \quad \text{if } i \neq 1$$

Thus,

$$Y_n = \begin{bmatrix}
+ & + & + & \cdots & + \\
- & - & & & \\
+ & - & & & \\
\vdots & & & & \\
\text{sgn}(-1)^{n-1} & - & \cdots & \cdots & -
\end{bmatrix}$$

Note that $Y_n$ is defined for $n = 2$ while $Z_{np}$ is not. Although $Z_{np}$ and $Y_n$ are very similar, our proofs that these are SAPs are simpler if each pattern is treated individually.

**Theorem 3.2.** The star sign pattern $Y_n$ ($n \geq 2$) is spectrally arbitrary.

**Proof.** Let $g(x) = x^n + \sum_{i=0}^{n-1} \mu_i x^i$ be any real monic polynomial of degree $n \geq 2$. Find positive $t$ greater than or equal to the absolute value of any zero of $g(x)$; for example, as in Lemma 2.5, pick $t = \max\{\sum_{i=0}^{n-1} |\mu_i|, 1\}$. Pick positive numbers $a_0, a_1, \ldots, a_{n-1}$ such that
First, pick the numbers so they satisfy the inequalities, then increase either \( a_1 \) or \( a_0 \) until equality in the second condition is satisfied. For each \( i \) with \( 1 \leq i \leq n-1 \), let

\[
c_i = \frac{(-1)^{i+1} g(-a_i)}{\prod_{j \neq i} (-a_i + a_j)}
\]

By Lemma 2.5, and Lemma 2.6 with \( q = 0 \) and \( p = n - 1 \geq 1 \) in the first statement, \( \text{sgn}(c_i) = + \).

Consider the following matrix having sign pattern \( Y_n \):

\[
M_n = \begin{bmatrix}
  a_0 & 1 & \cdots & 1 \\
  -c_1 & -a_1 & & \\
  c_2 & -a_2 & & \\
  \vdots & & & \\
  (-1)^{n-1}c_{n-1} & & & -a_{n-1}
\end{bmatrix}
\]

Note that \( a_i \) and \( c_i \) are positive for all \( i \), and that for the entries in \( M_n \), \( c_i \) is multiplied by \((-1)^i\). By Lemma 2.1, the characteristic polynomial of \( M_n \) is

\[
f_n(x) = (x - a_0) \prod_{i=1}^{n-1} (x + a_i) + \sum_{i=1}^{n-1} \left( (-1)^{i+1} c_i \prod_{j \neq i}^{n-1} (x + a_j) \right)
\]

This polynomial is monic of degree \( n \), and the coefficient of \( x^{n-1} \) is

\[-a_0 + \sum_{j=1}^{n-1} a_j = \mu_{n-1}\]

which is the coefficient of \( x^{n-1} \) in \( g(x) \). For each \( i \) with \( 1 \leq i \leq n-1 \),

\[
f_n(-a_i) = (-1)^{i+1} c_i \prod_{j \neq i}^{n-1} (-a_i + a_j) = g(-a_i)
\]

As in the proof of Theorem 3.1, \( f_n(x) - g(x) = 0 \). The characteristic polynomial of \( M_n \) is \( g(x) \) and \( M_n \) has sign pattern \( Y_n \). Thus, \( Y_n \) is a spectrally arbitrary star sign pattern.

Up to equivalence (as explained in Section 1), \( Y_2 \) is the unique spectrally arbitrary \( 2 \times 2 \) sign pattern, and \( Z_{31} \) and \( Y_3 \) are the only two minimal spectrally arbitrary \( 3 \times 3 \) tree sign patterns; see [1] and [3] (in which \( Z_{31} \) and \( Y_3 \) are denoted by \( \mathcal{F}_3 \) and \( \mathcal{W}_3 \), respectively).
4. Super-patterns of spectrally arbitrary star sign patterns

Some terminology is introduced for the statements in this section. Let $X = \{x_j \mid j \in J\}$ be a set of variables with finite index set $J$. Let $\Sigma_0(X) = 1$ and let $\Sigma_i(X)$ be the sum of the products of the elements of the $i$-element subsets of $X$ for each $i$ with $1 \leq i \leq |J|$. For each $j \in J$, let $X_j = X \setminus \{x_j\}$. For example,

$$\Sigma_1(X_j) = \sum_{k \in J \setminus \{j\}} x_k, \quad \Sigma_2(X_j) = \sum_{k, \ell \in J \setminus \{j\}} x_k x_\ell$$

**Lemma 4.1.** Let $n \geq 2$ and $X = \{x_1, x_2, \ldots, x_n\}$. For each $i$ with $1 \leq i \leq n$, if $j, k \in J$ and $j \neq k$, then

$$\Sigma_i(X_k) - \Sigma_i(X_j) = (x_j - x_k) \Sigma_{i-1}(X_j \cap X_k)$$

**Proof.**

$$\Sigma_i(X_k) = \Sigma_i(X_j \cap X_k) + x_j \Sigma_{i-1}(X_j \cap X_k)$$

$$\Sigma_i(X_j) = \Sigma_i(X_j \cap X_k) + x_k \Sigma_{i-1}(X_j \cap X_k)$$

and subtracting gives the result. $\Box$

**Lemma 4.2.** Let $n \geq 2$ and $X = \{x_1, x_2, \ldots, x_n\}$. If

$$A_n = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
\Sigma_1(X_1) & \Sigma_1(X_2) & \cdots & \Sigma_1(X_n) \\
\Sigma_2(X_1) & \Sigma_2(X_2) & \cdots & \Sigma_2(X_n) \\
\cdots & \cdots & \ddots & \cdots \\
\Sigma_{n-1}(X_1) & \Sigma_{n-1}(X_2) & \cdots & \Sigma_{n-1}(X_n)
\end{bmatrix}$$

then $\det(A_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j)$. 

**Proof.** (By induction.) For $n = 2$, $X = \{x_1, x_2\}$ and

$$\det(A_2) = \begin{vmatrix} 1 & 1 \\ x_2 & x_1 \end{vmatrix} = x_1 - x_2$$

Now suppose that the statement is true for all $m$ with $1 \leq m \leq n - 1$. Consider the matrix $A_n$. Subtracting the first column from each of the other columns and expanding about row 1, gives

$$\det(A_n) = \begin{vmatrix}
\Sigma_1(X_2) - \Sigma_1(X_1) & \Sigma_1(X_3) - \Sigma_1(X_1) & \cdots & \Sigma_1(X_n) - \Sigma_1(X_1) \\
\Sigma_2(X_2) - \Sigma_2(X_1) & \Sigma_2(X_3) - \Sigma_2(X_1) & \cdots & \Sigma_2(X_n) - \Sigma_2(X_1) \\
\vdots & \vdots & \ddots & \vdots \\
\Sigma_{n-1}(X_2) - \Sigma_{n-1}(X_1) & \Sigma_{n-1}(X_3) - \Sigma_{n-1}(X_1) & \cdots & \Sigma_{n-1}(X_n) - \Sigma_{n-1}(X_1)
\end{vmatrix}$$
by Lemma 4.1. Taking out common column factors gives

$$\det(A_n) = \left( \prod_{i=2}^{n} (x_i - x) \right) \begin{vmatrix} 1 & \Sigma_1(X_1 \cap X_2) & \Sigma_1(X_1 \cap X_3) & \cdots & \Sigma_1(X_1 \cap X_n) \\ \Sigma_2(X_1 \cap X_2) & 1 & \Sigma_2(X_1 \cap X_3) & \cdots & \Sigma_2(X_1 \cap X_n) \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \Sigma_{n-2}(X_1 \cap X_2) & \Sigma_{n-2}(X_1 \cap X_3) & \cdots & 1 & \Sigma_{n-2}(X_1 \cap X_n) \\ \Sigma_n(X_1 \cap X_2) & \Sigma_n(X_1 \cap X_3) & \cdots & \Sigma_n(X_1 \cap X_n) & 1 \end{vmatrix}$$

$$= \left( \prod_{i=2}^{n} (x_i - x) \right) \begin{vmatrix} 1 & \Sigma_1(W_2) & \Sigma_1(W_3) & \cdots & \Sigma_1(W_n) \\ \Sigma_2(W_2) & 1 & \Sigma_2(W_3) & \cdots & \Sigma_2(W_n) \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \Sigma_{n-2}(W_2) & \Sigma_{n-2}(W_3) & \cdots & 1 & \Sigma_{n-2}(W_n) \\ \Sigma_n(W_2) & \Sigma_n(W_3) & \cdots & \Sigma_n(W_n) & 1 \end{vmatrix}$$

where $W = \{x_2, x_3, \ldots, x_n\}$. Using the inductive hypothesis,

$$\det(A_n) = \prod_{i=2}^{n} (x_i - x) \prod_{2 \leq i < j \leq n} (x_i - x_j) = \prod_{1 \leq i < j \leq n} (x_i - x_j). \quad \square$$

A different proof of Lemma 4.2 can be obtained using the theory of alternants, see [8].

The following theorem and its proof using the implicit function theorem can be found in [1]; also see [3, Lemma 2.1].

**Theorem 4.3.** Let $S_n$ be an $n \times n$ sign pattern, and suppose that there exists a nilpotent matrix $M_n$ having sign pattern $S_n$ and at least $n$ non-zero entries, $m_{i_1,j_1}$, $m_{i_2,j_2}, \ldots, m_{i_n,j_n}$. Let $\mathcal{M}_n$ be the matrix obtained by replacing these entries in $M_n$ by variables $u_1, u_2, \ldots, u_n$, and let the characteristic polynomial of $\mathcal{M}_n$ be

$$f_n(x) = x^n - h_1 x^{n-1} + h_2 x^{n-2} + \cdots + (-1)^{n-1} h_{n-1} x + (-1)^n h_n$$

If the Jacobian

$$J = \frac{\partial(h_1, h_2, \ldots, h_n)}{\partial(u_1, u_2, \ldots, u_n)} = \det \left( \begin{bmatrix} \frac{\partial h_i}{\partial u_j} \end{bmatrix} \right)$$

is non-zero at $(u_1, u_2, \ldots, u_n) = (m_{i_1,j_1}, m_{i_2,j_2}, \ldots, m_{i_n,j_n})$, then every super-pattern of $S_n$ is spectrally arbitrary.
Theorem 4.4. Let \( n \geq 2 \) and \( S_n \) be a spectrally arbitrary star sign pattern. Then every super-pattern of \( S_n \) is spectrally arbitrary.

Proof. Let \( S_n \) be a spectrally arbitrary star sign pattern. Then, \( S_n \) is potentially nilpotent and so there is an \( n \times n \) nilpotent real matrix \( M_n \) (of the form given in Lemma 2.2) with sign pattern \( S_n \) where \( \alpha_i, \beta_i \neq 0 \) for \( i \geq 2 \) and \( \alpha_i \neq \alpha_j \) for \( i, j \geq 2 \) and \( i \neq j \) (by results in Section 2). Let

\[
\mathcal{M}_n = \begin{bmatrix}
\alpha_1 & 1 & 1 & \cdots & 1 \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
u_{n-1} & v_1 & v_2 & \cdots & v_{n-1}
\end{bmatrix}
\]

The conclusion will be shown to follow from an application of Theorem 4.3 with variables \( u_1, u_2, \ldots, u_{n-1}, v_{n-1} \). By Lemma 2.1, the characteristic polynomial of \( \mathcal{M}_n \) is

\[
f_n(x) = (x - \alpha_1) \prod_{j=1}^{n-1} (x - v_j) - \sum_{i=1}^{n-1} \left( u_i \prod_{j \neq i}^{n-1} (x - v_j) \right)
\]

Letting

\[
f_n(x) = x^n - h_1 x^{n-1} + h_2 x^{n-2} + \cdots + (-1)^{n-1} h_{n-1} x + (-1)^n h_n
\]

and \( X = \{v_1, v_2, \ldots, v_{n-1}\} \), then

\[
h_1 = \alpha_1 + \sum_{i=1}^{n-1} v_i = \Sigma_1(X \cup \{\alpha_1\})
\]

\[
h_2 = \alpha_1 \sum_{i=1}^{n-1} v_i + \sum_{1 \leq i_1 < i_2 \leq n-1} v_{i_1} v_{i_2} - \sum_{i=1}^{n-1} u_i
\]

\[
= \Sigma_2(X \cup \{\alpha_1\}) - \sum_{i=1}^{n-1} u_i \Sigma_0(X_i)
\]

\[
h_3 = \alpha_1 \sum_{1 \leq i_1 < i_2 < i_3 \leq n-1} v_{i_1} v_{i_2} v_{i_3}
\]

\[
- \sum_{i=1}^{n-1} \left( u_i \sum_{1 \leq i_1 < i_2 \leq n-1} v_{i_1} v_{i_2} \right)
\]

\[
= \Sigma_3(X \cup \{\alpha_1\}) - \sum_{i=1}^{n-1} u_i \Sigma_1(X_i)
\]
and in general for \( \ell \geq 2 \),
\[
h_\ell = \Sigma_\ell (X \cup \{a_1\}) - \sum_{i=1}^{n-1} u_i \Sigma_{\ell-2}(X_i)
\]
Taking the partial derivatives with respect to \( v_{n-1} \) and \( u_i \) for \( 1 \leq i \leq n-1 \):
\[
\frac{\partial h_1}{\partial v_{n-1}} = 1 \quad \frac{\partial h_1}{\partial u_i} = 0 \quad \frac{\partial h_\ell}{\partial u_i} = -\Sigma_{\ell-2}(X_i) \quad (\ell \geq 2)
\]
The partial derivatives \( \frac{\partial h_\ell}{\partial v_{n-1}} \) are not computed for \( \ell \geq 2 \) because an examination of the following determinant shows that these terms do not appear in the calculation of the Jacobian (each such term is replaced by * in \( J \) below). Now, let \( u_n = v_{n-1} \). Then the Jacobian
\[
J = \left[ \frac{\partial (h_1, h_2, \ldots, h_n)}{\partial (u_1, u_2, \ldots, u_n)} \right]
\]
\[
= \begin{vmatrix}
0 & 0 & \cdots & 0 & 1 \\
-1 & -1 & \cdots & -1 & * \\
-\Sigma_1(X_1) & -\Sigma_1(X_2) & \cdots & -\Sigma_1(X_{n-1}) & * \\
-\Sigma_2(X_1) & -\Sigma_2(X_2) & \cdots & -\Sigma_2(X_{n-1}) & * \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-\Sigma_{n-2}(X_1) & -\Sigma_{n-2}(X_2) & \cdots & -\Sigma_{n-2}(X_{n-1}) & *
\end{vmatrix}
\]
\[
= (-1)^{2n-2} \begin{vmatrix}
1 & 1 & \cdots & 1 \\
\Sigma_1(X_1) & \Sigma_1(X_2) & \cdots & \Sigma_1(X_{n-1}) \\
\Sigma_2(X_1) & \Sigma_2(X_2) & \cdots & \Sigma_2(X_{n-1}) \\
\vdots & \vdots & \ddots & \vdots \\
\Sigma_{n-2}(X_1) & \Sigma_{n-2}(X_2) & \cdots & \Sigma_{n-2}(X_{n-1})
\end{vmatrix}
\]
\[
= \prod_{1 \leq i < j \leq n-1} (v_i - v_j)
\]
by Lemma 4.2. Since the \( a_i \)s are distinct for \( 2 \leq i \leq n \), \( J \) is non-zero at
\[
(v_1, v_2, \ldots, v_{n-1}, u_1, u_2, \ldots, u_{n-1}) = (a_2, a_3, \ldots, a_n, \beta_2, \beta_3, \ldots, \beta_n).
\]
Therefore, by Theorem 4.3, every super-pattern of \( S_n \) is spectrally arbitrary. \( \square \)

**Corollary 4.5.** Every super-pattern of \( Y_n \) or \( Z_{np} \) (for appropriate \( n \) and \( p \)) is spectrally arbitrary, and thus inertially arbitrary.

Let \( Z_{np}^+ \) and \( Z_{np}^- \) be the super-patterns of \( Z_{np} \) obtained by replacing the 0 in \( z_{11} \) by \( + \) and \( - \), respectively. By the above corollary, these star sign patterns are SAPs and IAPs.
5. Potentially stable and inertially arbitrary star sign patterns

If sign pattern \( S_n \) is inertially arbitrary, then both \( S_n \) and \(-S_n\) are potentially stable. Such sign patterns are considered in Theorem 5.2 below. The proof of this theorem relies on the next result, which follows directly from theorems proved in [5, Theorems 3.5 and 4.2].

**Theorem 5.1.** Let \( n \geq 2 \) and \( S_n \) be a potentially stable star sign pattern where \( s_{12}, s_{13}, \ldots, s_{1n} \) are + and \( s_{22}, s_{33}, \ldots, s_{nn} \) are non-zero. If among \( s_{22}, s_{33}, \ldots, s_{nn} \) exactly \( q \) terms are +, say \( s_{i_1j_1}, s_{i_2j_2}, \ldots, s_{i_qj_q} \) are +, then among \( s_{i_1j_1}, s_{i_2j_2}, \ldots, s_{i_qj_q} \) exactly \( \left\lfloor \frac{q}{2} \right\rfloor \) terms are +.

**Theorem 5.2.** Let \( n \geq 2 \) and \( S_n \) be a star sign pattern where \( s_{ii} \) is non-zero for any leaf \( i \). If \( S_n \) and \(-S_n\) are both potentially stable, then \( S_n \) is equivalent to one of \( Y_n, Z^+, Z^- \) (for appropriate \( p \)).

**Proof.** Let \( n \geq 2 \) and \( S_n \) be a star sign pattern where \( s_{22}, s_{33}, \ldots, s_{nn} \) are non-zero and \( S_n \) and \(-S_n\) are both potentially stable. For all \( i \) such that \( 2 \leq i \leq n, s_{ii} \) and \( s_{1i} \) are non-zero, otherwise \( S_n \) would not be a star sign pattern.

**Case 1:** Among \( s_{22}, s_{33}, \ldots, s_{nn} \) there are both + and - entries. This implies \( n \geq 3 \). By permutation similarity, assume that \( s_{22}, s_{33}, \ldots, s_{p+1,p+1} \) are + and \( s_{p+2,p+2}, s_{p+3,p+3}, \ldots, s_{nn} \) are + for some \( p \) such that \( 1 \leq p \leq n-2 \). By Theorem 5.1, there are exactly \( \left\lfloor \frac{n-p-1}{2} \right\rfloor \) positive entries, and thus exactly \( \left\lceil \frac{n-p-1}{2} \right\rceil \) negative entries, among \( s_{p+2,1}, s_{p+3,1}, \ldots, s_{n1} \). This is exactly the number of positive and negative entries among \( z_{p+2,1}, z_{p+3,1}, \ldots, z_{n1} \) in \( Z_{np} \). Since the off-diagonal terms in a matrix having a star sign pattern enter into its characteristic polynomial only as products, \(-S_n\) is equivalent to the pattern obtained from \( S_n \) by taking the negative of the main diagonal. So, similarly, there are exactly \( \left\lceil \frac{n}{2} \right\rceil \) positive entries among \( s_{21}, s_{31}, \ldots, s_{p+1,1} \) and thus \( \left\lfloor \frac{n}{2} \right\rfloor \) negative entries. This is exactly the number of positive and negative entries among \( z_{21}, z_{31}, \ldots, z_{p+1,1} \) in \( Z_{np} \). So, by permutation similarity, \( s_{ij} = z_{ij} \) in \( Z_{np} \) for all \( (i, j) \neq (1, 1) \). Therefore, \( S_n \) is equivalent to one of \( Z_{np}, Z^+, Z^- \) (depending on the sign of \( s_{11} \)).

**Case 2:** The entries \( s_{22}, s_{33}, \ldots, s_{nn} \) are all the same sign. Since \(-S_n\) is equivalent to the star sign pattern obtained by taking the negative of the main diagonal in \( S_n \), assume \( s_{22}, s_{33}, \ldots, s_{nn} \) are all -. Thus, \( s_{11} \) is +, otherwise \(-S_n\) would have a strictly positive main diagonal. Furthermore, similarly to Case 1, there are exactly \( \left\lceil \frac{n-1}{2} \right\rceil \) positive and \( \left\lfloor \frac{n-1}{2} \right\rfloor \) negative entries among \( s_{21}, s_{31}, \ldots, s_{n1} \). This is exactly the number of positive and negative entries among \( y_{21}, y_{31}, \ldots, y_{n1} \) in \( Y_n \). Therefore, \( S_n \) is equivalent to \( Y_n \). \( \square \)
Note that Theorem 5.2 is sufficient for our purposes, since IAPs and SAPs are subclasses of the patterns $S_n$ for which $S_n$ and $-S_n$ are both potentially stable. However, using further results from [5], a characterization of all star sign patterns $S_n$ such that $S_n$ and $-S_n$ are both potentially stable can be obtained.

**Theorem 5.3.** Let \( n \geq 2 \) and \( S_n \) be an inertially arbitrary star sign pattern. Then \( S_n \) is equivalent to one of \( Y_n \), \( Z_{np} \), \( Z_{np}^+ \) or \( Z_{np}^- \) (for appropriate \( p \)). Furthermore, \( Y_n \) and \( Z_{np} \) are minimal inertially arbitrary sign patterns and the only minimal inertially arbitrary star sign patterns.

**Proof.** Let \( S_n \) be an inertially arbitrary star sign pattern. By Lemma 2.2, the entries \( s_{22}, s_{33}, \ldots, s_{nn} \) are non-zero, thus the first statement follows directly from Theorem 5.2. Clearly, \( Z_{np}^+ \) and \( Z_{np}^- \) are not minimal. As in Section 2, if \( s_{11} \) or \( s_{1i} \) is replaced with 0 where \( 2 \leq i \leq n \), any matrix with the resulting sign pattern has an eigenvalue with sign equal to \( s_{1i} \) and so the sign pattern is not an IAP. If \( s_{1i} \) is replaced with 0 where \( 2 \leq i \leq n \), any matrix with the resulting sign pattern has the sign of its determinant fixed, and so the sign pattern is not an IAP. In \( Z_{np} \), the entry \( z_{11} \) is equal to 0, so \( Z_{np} \) is minimal. In \( Y_n \), the entry \( y_{11} \) is the only positive entry on the main diagonal, and so cannot be replaced by 0, so \( Y_n \) is minimal. \( \square \)

Although the sign patterns \( Y_n \) and \( Z_{np} \) are minimal inertially (and thus, spectrally) arbitrary star sign patterns, there are proper sub-patterns of these that are potentially nilpotent or potentially stable. For example,

\[
\begin{bmatrix}
  0 & + & + \\
  + & 0 & 0 \\
  - & 0 & 0 
\end{bmatrix}
\]

is potentially nilpotent and

\[
\begin{bmatrix}
  0 & + & + \\
  - & - & 0 \\
  - & 0 & 0 
\end{bmatrix}
\]

is potentially stable.

### 6. Potentially nilpotent star sign patterns

**Theorem 6.1.** Let \( n \geq 2 \) and \( S_n = [s_{ij}] \) be a star sign pattern. If \( S_n \) is potentially nilpotent and has \( s_{1i} \neq 0 \) for any leaf \( i \), then, up to equivalence, \( S_n \) is one of \( Y_n \), \( Z_{np} \), \( Z_{np}^+ \) or \( Z_{np}^- \) (for appropriate \( p \)).
Proof. Let \( n \geq 2 \) and \( S_n \) be a potentially nilpotent star sign pattern with no zeroes among \( s_{22}, s_{33}, \ldots, s_{nn} \). Let \( M_n \), as in Lemma 2.2, be a nilpotent matrix having sign pattern \( S_n \). Thus, by Lemma 2.4, for \( i \geq 2 \), the \( \alpha_i \) entries are distinct. By permutation similarity, it can be assumed that \( \alpha_2 < \alpha_3 < \cdots < \alpha_n \).

Case 1: Among \( \alpha_2, \alpha_3, \ldots, \alpha_n \), there are both positive and negative entries. This implies that \( n \geq 3 \). Let \( p \) and \( q \) be the number of negative and positive entries, respectively (and note that \( 1 \leq p, q \leq n - 2 \) and that \( p + q + 1 = n \)). Then
\[
\alpha_2 < \alpha_3 < \cdots < \alpha_{p+1} < 0 < \alpha_{p+2} < \alpha_{p+3} < \cdots < \alpha_n
\]

Relabel \( \alpha_2, \alpha_3, \ldots, \alpha_n, \beta_2, \beta_3, \ldots, \beta_n \) to give
\[
M_n = \begin{bmatrix}
\alpha_1 & 1 & 1 & \cdots & \cdots & \cdots & 1 \\
\gamma_2 & -a_1 \\
\gamma_2 & -a_2 \\
\vdots & \ddots \\
\gamma_p & -a_p & b_1 \\
\delta_1 & & & \ddots \\
\delta_2 & & b_2 & \ddots \\
\vdots & & & \ddots & \\
\delta_q & & & & & b_q
\end{bmatrix}
\]
where \( 0 < a_p < a_{p-1} < \cdots < a_1 \) and \( 0 < b_1 < b_2 \) \( \cdots < b_q \). By Lemma 2.1, the characteristic polynomial of \( M_n \) is
\[
f_n(x) = (x - \alpha_1) \prod_{j=1}^{p} (x + a_j) \prod_{k=1}^{q} (x - b_k)
\]
\[
= \sum_{i=1}^{p} \gamma_i \prod_{1 \leq j \leq p, j \neq i} (x + a_j) \prod_{k=1}^{q} (x - b_k)
\]
\[
= \sum_{i=1}^{q} \delta_i \prod_{1 \leq j \leq q, j \neq i} (x + a_j) \prod_{k=1}^{p} (x - b_k)
\]

Since \( M_n \) is nilpotent, \( f_n(x) = x^n \). So, for each \( i \) with \( 1 \leq i \leq p \),
\[
(-a_i)^n = -\gamma_i \prod_{1 \leq j \leq p, j \neq i} (-a_i + a_j) \prod_{k=1}^{q} (-a_i - b_k)
\]
By Lemma 2.6, \( \text{sgn}(\gamma_i) = \text{sgn}(-1)^{n-1-(p+q-i)} = \text{sgn}(-1)^i \) (since \( p + q + 1 = n \)). Moreover, for each \( i \) with \( 1 \leq i \leq q \),
which implies, by Lemma 2.6, that \( \text{sgn}(\delta_i) = (-1)^{q-i+1} \). So, for \((i, j) \neq (1, 1)\), the \((i, j)\)th entry of \(M_n\) has sign equal to \(z_{ij}\) in \(Z_{np}\). Thus, \(S_n\) is equivalent to one of \(Z_{np}^+\), \(Z_{np}^-\) or \(Z_{np}^{\pm}\).

Case 2: For \(i \geq 2\), the \(\alpha_i\) entries all have the same sign; without loss of generality, assume that they are all negative (if \(M_n\) is nilpotent, so is \(-M_n\)). Order the leaf entries so that \(\alpha_2 < \alpha_3 < \cdots < \alpha_n\) and then relabel the matrix to give

\[
M_n = \begin{bmatrix}
\alpha_1 & 1 & 1 & \cdots & 1 \\
\gamma_1 & -a_1 & & & \\
\gamma_2 & -a_2 & & \\
\vdots & & \ddots & & \\
\gamma_{n-1} & & \cdots & -a_{n-1}
\end{bmatrix}
\]

Since \(M_n\) is nilpotent, \(\alpha_1 = \sum_{j=1}^{n-1} a_j\) is positive. Furthermore, by Lemma 2.1, the characteristic polynomial of \(M_n\) is

\[
f_n(x) = (x - \alpha_1) \prod_{i=1}^{n-1} (x + a_i) - \sum_{i=1}^{n-1} (x + a_i) \prod_{1 \leq j < m \leq n - 1} (x - a_i + a_j)
\]

Since \(M_n\) is nilpotent, \(f_n(x) = x^n\), so for each \(i\) with \(1 \leq i \leq n - 1\),

\[
(-a_i)^n = -\gamma_i \prod_{1 \leq j < m \leq n - 1} (-a_i + a_j)
\]

implying that \(\gamma_i\) has sign \(\text{sgn}(-1)^i\). So, by permutation similarity, for all \((i, j)\), the sign of the \((i, j)\)th entry of \(M_n\) is equal to \(\gamma_{ij}\) in \(Y_n\). Therefore, \(S_n\) is equivalent to \(Y_n\). \(\square\)

Theorem 6.1, Corollary 4.5 and results of [7], which are introduced next, can be used to characterize potentially nilpotent star sign patterns. Let \(n \geq 2\) and \(S_n\) be a star sign pattern. By permutation similarity, it can be assumed that \(s_{22}, s_{33}, \ldots, s_{nn}\) are non-zero and \(s_{m+1,m+1}, s_{m+2,m+2}, \ldots, s_{nn}\) are zero (either collection could be empty). Let \(R_1\) be the \(m \times m\) star sign pattern obtained by deleting the rows and columns numbered \(m + 1, m + 2, \ldots, n\) in \(S_n\). Let \(R_2\) be the \((n - m + 1) \times (n - m + 1)\) star sign pattern obtained by deleting rows and columns numbered \(2, 3, \ldots, m\) in \(S_n\) and replacing \(s_{11}\) with 0. For example, if
Theorem 6.2 [7, Theorems 2 and 3]. Let \( n \geq 2 \) and \( S_n \) be a star sign pattern. Then \( S_n \) is potentially nilpotent if and only if \( R_1 \) and \( R_2 \) (as above) obtained from \( S_n \) are potentially nilpotent. In particular, \( R_2 \) is potentially nilpotent if and only if there is at least one positive and one negative entry among \( s_{m+1,1}, s_{m+2,1}, \ldots, s_{n1} \).

Theorem 6.1 and Corollary 4.5 give necessary and sufficient conditions for the star sign pattern \( R_1 \) to be potentially nilpotent if \( R_1 \) has order at least 2. Theorem 6.2 gives necessary and sufficient conditions for the star sign pattern \( R_2 \) to be potentially nilpotent if the order of \( R_2 \) is at least 2; moreover, the only \( 1 \times 1 \) potentially nilpotent sign pattern is \( S_1 = [0] \).

**Theorem 6.3.** Let \( n \geq 2 \) and \( S_n \) be a star sign pattern. Then \( S_n \) is potentially nilpotent if and only if, up to equivalence,

(a) \( S_n \) is one of \( Y_n, Z_{np}, Z_{np}^+, Z_{np}^- \) (for appropriate \( p \)); or,
(b) for some \( m \) such that \( 1 \leq m \leq n-2 \),

\[
S_n = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}
\]
where

\[ S_{12} = \begin{bmatrix}
+ & + & \cdots & + \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix} \]

is \( m \times (n - m) \);

\[ S_{21} = \begin{bmatrix}
{s_{m+1,1}} & 0 & 0 & \cdots & 0 \\
{s_{m+2,1}} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
{s_{n1}} & 0 & 0 & \cdots & 0
\end{bmatrix} \]

is \((n - m) \times m\) and has both positive and negative entries among \( s_{m+1,1}, s_{m+2,1}, \ldots, s_{n1} \); \( S_{22} = [0]_{(n-m) \times (n-m)} \); and,

(i) if \( m = 1 \), then \( S_{11} = [0] \), or

(ii) if \( m \geq 2 \), then \( S_{11} \) is one of \( Y_{m, Z_{np}}, Z_{np}^{+}, Z_{np}^{-} \) (for appropriate \( p \)).

**Theorem 6.4.** Let \( n \geq 2 \) and \( S_n \) be a star sign pattern that is both potentially nilpotent and potentially stable. Then \( S_n \) has \( s_{ii} \neq 0 \) for any leaf \( i \). In particular, \( S_n \) is equivalent to one of \( Y_{n, Z_{np}}, Z_{np}^{+}, Z_{np}^{-} \) (for appropriate \( p \)).

**Proof.** Suppose \( n \geq 2 \) and \( S_n \) is a potentially nilpotent and potentially stable star sign pattern. Let \( m \) be the number of non-zero entries among \( s_{22}, s_{33}, \ldots, s_{nn} \). If \( m \leq n - 3 \), then there are at least two zeroes among \( s_{22}, s_{33}, \ldots, s_{nn} \), so any matrix having sign pattern \( S_n \) has determinant equal to zero. Thus, \( m \geq n - 2 \), since \( S_n \) is potentially stable. If \( m = n - 2 \), then there is exactly one zero among \( s_{22}, s_{33}, \ldots, s_{nn} \), implying that any matrix having sign pattern \( S_n \) has a non-zero determinant. Thus, since \( S_n \) is potentially nilpotent, \( m = n - 1 \). So, \( s_{ii} \neq 0 \) for all \( i \geq 2 \). Therefore, by Theorem 6.3, \( S_n \) is equivalent to one of \( Y_{n, Z_{np}}, Z_{np}^{+}, Z_{np}^{-} \) (for appropriate \( p \)). \( \square \)

7. Summary

The following theorem summarizes our results for star sign patterns.

**Theorem 7.1.** If \( n \geq 2 \) and \( S_n \) is a star sign pattern, then the following are equivalent:

1. \( S_n \) is equivalent to one of \( Y_{n, Z_{np}}, Z_{np}^{+}, Z_{np}^{-} \) (for appropriate \( p \)).
2. \( S_n \) is spectrally arbitrary.
3. $S_n$ is inertially arbitrary.
4. $S_n$ is potentially nilpotent and has a loop at each of the $n - 1$ leaves in its graph.
5. $S_n$ and $-S_n$ are both potentially stable and $S_n$ has a loop at each of the $n - 1$ leaves in its graph.
6. $S_n$ is potentially nilpotent and potentially stable.

The equivalences in Theorem 7.1 are illustrated in the strongly connected directed graph given above, with the implications following from the definitions of the appropriate terms along with previous theorems, corollaries and lemmas given by number.

The equivalence of (2) and (3) holds for any $3 \times 3$ sign pattern; see [3]. However, in general (3) does not imply (2), as illustrated by an example with $n = 4$ in [4] where the graph of the sign pattern is not a tree. The same example shows that (6) does not necessarily imply (2) for general sign patterns. However, for tree sign patterns with $n \geq 4$, it is unknown whether (3) implies (2), or whether (6) implies (2) or (3).
Acknowledgment

The authors thank D.D. Olesky for a careful reading of this manuscript and T. Britz for discussions at the beginning of this project.

References