Time Dependent Anomalous Diffusion for Flow in Multi-fractal Porous Media *

Frederico Furtado **, James Glimm ***, Brent Lindquist **, Felipe Pereira **, and Qiang Zhang **

Department of Applied Mathematics and Statistics, University at Stony Brook, Stony Brook, NY 11794-3600

1 Introduction

Macroscopic flow computations are used in the modeling of most practical porous media flow studies. These computations require macroscopic or effective flow parameters, describing effective fluid behavior over an integrated range of shorter length scales. This requirement is the primary reason for solving the scale up problem, which is that of determining or extrapolating such effective parameters from data measured at smaller length scales. Multi-length scale characterization of the medium heterogeneity is needed to solve this scale up problem. Such a characterization is usually missing or incomplete. Thus a second value of solving the scale up problem is to specify a minimum level of knowledge of porous media heterogeneity required in order to narrow the range of uncertainty in flow computations associated with an incomplete description of the media. We believe a solution of the scale up problem, adequate for these two purposes, is possible, and is one of the goals in our current investigations.

We concentrate on the effective diffusivity as a macroscopic model parameter describing the growth rate of a fluid mixing zone induced by a heterogeneous permeability field. The effective diffusion process can be non-Fickian (anomalous) for two reasons: multi-fractal rock statistics producing non-Fickian transient effects, or fractal rock statistics with slowly decaying correlations producing a non-Fickian, steady state response. The present article analyses the transient effects as a cause of non-Fickian diffusion, while earlier studies, [AHLL], [FGLP1], [FGLP2] and [GS] concerned the steady state response to fractal statistics. For the transient response work, we follow the theoretical analysis of [Z].

* Support from the computational facilities of the Numerical Aerodynamics Simulations Systems Division, NASA Ames Research Center, Moffett Field and of the Engineering Physics and Math Division, Oak Ridge National Laboratory is gratefully acknowledged.

** Supported by the Applied Mathematics Subprogram of the U.S. Department of Energy DE-FG02-90ER25084.

*** Supported by the Applied Mathematics Subprogram of the U.S. Department of Energy DE-FG02-90ER25084 and the National Science Foundation, grant DMS-89018844.
We define multi-fractal behavior as the modification of self-similar (fractal) behavior to account for different events occurring with different probabilities on different length scales. In section 2 we demonstrate the inclusion of this behavior in our earlier fractal models by considering $\xi = \log(\text{permeability})$, and its covariance function

$$C(x,y) \equiv <\xi(x)\xi(y)> = C(r), \quad r \equiv |x-y|,$$

(1)

which we assume depends only on separation distance $r$ (i.e., we assume stationarity and isotropy of the field $\xi$).

We note that further generalizations of our hypotheses are important, and will be addressed in later papers in this series. In order to understand the importance of transverse flow in the mixing process we will study the fluid flow in partially layered media. The inclusion of bimodal geology, such as shale barriers, would require the addition of Ising- or $\phi^4$-model statistics [GJ] to the Gaussian models we have used. The influence of curvilinear flow geometry is very important and previous work of the authors [GILMY] should be extended to the present context. Finally nonlinear flow physics (immiscible flow) should also be considered. The influence of microscopic mixing (molecular or capillary diffusion) and the interaction between macroscopic and microscopic diffusion (Taylor diffusion) appear to have smaller effects in comparison with the above list.

2 Multifractal Diffusion

We begin from a microscopic, linear transport equation coupled to a random velocity field,

$$s_t + \mathbf{v} \cdot \nabla s = 0 .$$

(2)

The variable $s$ denotes the volume fraction (saturation) of a tagged fluid displacing its untagged (saturation $1-s$) counterpart. We take the velocity field, $\mathbf{v}$, to be a random field, obtained by solution of Darcy’s law

$$\mathbf{v} = -\frac{\mathbf{K} e^\xi}{\mu} \nabla P ,$$

(3)

and the incompressibility condition,

$$\nabla \cdot \mathbf{v} = 0 .$$

(4)

In (3), $\mu$ is the viscosity of the fluid, $P$ is the pressure, $\mathbf{K} e^\xi$ is the rock permeability, which we assume to be a scalar field, $\mathbf{K}$ is constant, and $\xi$ is a random field, with stationary Gaussian statistics and mean value $<\xi> = 0$, describing the statistical variation of the rock permeability.

We choose boundary conditions for (2) – (4) so that the dominant flow is in the $y$-direction. Expanding (3) in powers of $\xi$, we obtain, to first order

$$\mathbf{v} = \mathbf{v}_0 + \delta \mathbf{v} , \quad \delta \mathbf{v} = \xi \mathbf{v}_0 - v_0 \nabla \left( G * \frac{\partial \xi}{\partial y} \right) ,$$

(5)
where \( v_0 = v_0 \mathbf{e}_y \) is the unperturbed, constant, \( y \)-direction flow, and \( G(x, x') \) is the Green’s function for the laplacian
\[
\Delta G(x, x') = \delta(x - x') .
\]
Thus, to first order in \( \xi \), the velocity field is also Gaussian. However, with boundary conditions imposed on a finite length domain in the \( y \)-direction, \( \delta \mathbf{v} \) will not satisfy stationary statistics, even if \( \xi \) does. We, therefore, simplify our formulas by assuming that the boundaries are at infinity, in which case \( \delta \mathbf{v} \) also satisfies stationary statistics. This first order theory will, of course, be exact in the limit of weak variation in the heterogeneity \( \xi \). One of the conclusions of our numerical studies is that the formulas we have derived for mixing length exponents in this weak heterogeneity limit are valid up to heterogeneities of moderate strength.

Flow under (2) - (4) results in macroscopic mixing induced by the heterogeneity in the rock permeability (and hence in \( \mathbf{v} \)). Assuming dominant flow in the \( y \)-direction, using (5), performing an ensemble average over all rock realizations having the same statistics and a spatial average over directions transverse to the \( y \)-direction, the resultant averaged saturation \( \langle s \rangle \) satisfies [D], [Z] a 1-D macroscopic advection-diffusion equation
\[
\frac{\partial \langle s \rangle}{\partial t} + v_0 \frac{\partial \langle s \rangle}{\partial y} = D(t) \frac{\partial^2 \langle s \rangle}{\partial y^2} ,
\]
where \( D(t) \) is an effective, time dependent viscosity coefficient. The derivation of (6) from (2) - (5) is exact to second order in the perturbation expansion in \( \xi \).

Under the above considerations, the time dependent, effective diffusion parameter \( D(t) \) can be shown [D], [Z] to be given, to leading order, by
\[
D(t) = \int_0^t < \delta \mathbf{v}(x - \sigma \mathbf{v}_0) \cdot \delta \mathbf{v}(x) > d\sigma ,
\]
where the integral is along backward streamlines of the unperturbed velocity field \( \mathbf{v}_0 \).

Consider first the time-asymptotic behavior \( D_\infty \equiv \lim_{t \to \infty} D(t) \). If \( D_\infty \) is finite, the asymptotic large time diffusion process is Fickian (normal), whereas if \( D_\infty = \infty \), the asymptotic large time diffusion process is non-Fickian (anomalous) [Z]. An examination of (5) to lowest order in \( \xi \), shows that the diffusion will be anomalous if and only if the backward streamline integral, analogous to (7), for the rock correlation \( \langle \xi \xi \rangle \) diverges.

To analyze finite time (transient) behavior, assuming stationary isotropic statistics, we express the correlation functions \( \langle \xi \xi \rangle \) and \( \langle \delta \mathbf{v} \cdot \delta \mathbf{v} \rangle \) (along streamlines) in multi-fractal form [Z]
\[
\langle \xi(0) \xi(r) \rangle = b(r) r^{-\beta(r)} , \quad \langle \delta \mathbf{v}(0) \cdot \delta \mathbf{v}(r) \rangle = a(r) r^{-\rho(r)} ,
\]
where \( \beta(r) \) (similarly \( \rho \) ) is the slope and \( b(r) \) (similarly \( a(r) \) ) is the intercept of the tangent line \( \tau(r) \) of the graph \( \ln(\langle \xi \xi \rangle) \) (similarly \( \ln(\langle \delta \mathbf{v} \cdot \delta \mathbf{v} \rangle) \)) versus \( \ln(r) \). This multi-fractal representation is useful when \( \beta(r) \) and \( \rho(r) \) are \textit{slowly}
varying functions, which we now assume. The effective saturation variable \( s \) develops a mixing length \( l(t) \) which can also be represented in multi-fractal form,

\[
l(t) \sim t^{\gamma(t)}, \quad \gamma(t) = \frac{1}{2} \frac{\partial \ln D(t)}{\partial \ln(t)} + \frac{1}{2},
\]

where, for slowly varying \( \beta \) (and \( \rho \)),

\[
\gamma(t) = \max \left\{ \frac{1}{2}, 1 - \frac{\beta(t)}{2} \right\}.
\]

Finally, for initial growth behavior, assuming the \( \xi \) correlation function is non-singular at short distances, the initial mixing length exponent has been determined [Z] to be \( \gamma(t = 0) = 1 \). Thus we see that, assuming the flow is asymptotically Fickian, \( \gamma_{\infty} = \lim_{t \to \infty} \gamma(t) = 1/2 \), the transients in the mixing process are necessarily non-Fickian and that the multi-fractal theory applies to the transient flow.

3 Computational Results

We report on two dimensional computational studies, using the front tracking method [GIMM] [GLMP] for high resolution of the mixing zone, for three cases of rock permeability exponent: \( \beta = 0.5, \beta = 1.25 \) and \( \beta = \infty \).

For each case, specific statistical realizations of the rock heterogeneity fields were generated by convolution of an independent Gaussian random field (white noise) with a weight function [FGLP1]. For \( 0 < \beta < 2 \), the weight function has a power law decay, while in the case \( \beta = \infty \) the weight function is a \( \delta \) function, which we smear out over one mesh block. We note that this numerical rock permeability field, \( \xi \), is not purely fractal, but is cut off at short distance by the effect of finite grid spacing. For each separate value of \( \beta \), the results discussed below are obtained from averaging over a number of realizations generated on grids of the same size. The number of realizations and grid sizes used for each value of \( \beta \) in the studies discussed below are given in Table 1.

For a given realization of the permeability field, the magnitude of the heterogeneity can be quantified by the coefficient of variation

\[
C_v \equiv \frac{\sigma'}{<K'>},
\]

where \( \sigma' \) and \( <K'> \) are, respectively, the standard deviation and mean of the permeability values in the given realization. The average coefficient of variation obtained from all realizations generated for a given \( \beta \) and grid size is given in column four of Table 1. The uncertainty quoted for each average indicates the range of coefficients of variation found in each of the individual realizations which contributed to the average.

In Fig. 1 we plot the numerically computed \( \ln(<K^\xi, \bar{K}^\xi>) \) correlation as a function of \( \ln(t) \) for the slowest decaying correlation fields, \( \beta = 0.5 \) having average \( C_v = 0.19 \). The correlation displays pure fractal behavior over a distance
Table 1. Number of realizations, grid sizes and coefficient of variations used for each value of $\beta$

of about 55 mesh blocks. The slope of this curve is $0.47$, in $6\%$ agreement with the theoretical decay exponent. Beyond this distance (not shown), the correlation falls off more rapidly and is not fractal. We believe that lack of statistical convergence is the cause of such large distance behavior, since fewer statistics are accumulated in a finite size domain when computing the correlation for large $r$ separation than for short separations.

Fig. 1. Correlation function $\ln(<K^{\ell(r)}K^{\ell(0)}>)$ versus $\ln(r)$ for $\beta = 0.5$, $C_\gamma = 0.19$.

Because of their rapid decay, the permeability correlation functions for $\beta =$
1.25 and $\beta = \infty$ are harder to measure on grids of finite spacing. For $\beta = \infty$, the measured correlation function is consistent with a numerical $\delta$ function, dropping to 1% of its zero separation value in one mesh block distance. The correlation function for $\beta = 1.25$ drops to 1% of its zero separation value over four mesh blocks. The zero separation value is based on a subgrid resolution distance corresponding to $1/32$ of a mesh block width. This was determined from local mesh refinement used in generating the permeability field.

For fixed values of $\beta$, $C_v$ and grid size, the mixing length $l(t)$ was determined from the ensemble and spatially transverse averaged profiles of $< s >$. At a fixed $t$, the profile was fit with a complementary error function erfc containing two free parameters, a mean profile travel distance $L$ (it is convenient to use the travel distance $L$ rather than the time $t$), and a variance, whose square root determines the mixing length $l(L)$. In Fig. 2 we display the averaged saturation profiles $< s >$ and fitted erfc profiles, at similar $L$ values, as a function of $C_v$, obtained for $\beta = \infty$ fields. For the largest $C_v$ (i.e. the greatest heterogeneity strength) the profiles begin to deviate from an erfc. It is not clear whether averaging over more realizations would be sufficient to correct this deviation at large heterogeneity strengths.

Fig. 2. The averaged saturation profiles at approximately fixed average flow distance $L$ for flows in $\beta = \infty$ permeability fields characterized by 4 different coefficients of variation: (a) 0.18 ± 0.005, (b) 0.38 ± 0.01, (c) 0.69 ± 0.02 and (d) 0.96 ± 0.04. Superimposed upon each is the best fit erfc (smooth curve).
Expressing the asymptotic mixing length behavior \( l_\infty \equiv l(L \to \infty) \) in pure fractal form

\[ l_\infty = a_\infty \xi^{\gamma_\infty}, \]

it can be shown that, in the small \( \xi \) limit, the fractal exponent \( \gamma_\infty \) is independent of \( C_v \), and the scale factor \( a_\infty \) varies linearly with \( C_v \). The results of mixing length exponent \( \gamma \) obtained for \( \beta = \infty \) from four separate sets of runs (corresponding to rows 3 to 6 of Table 1) with differing \( C_v \) are shown in Table 2. The results are all within 2\% agreement of the asymptotic prediction of \( \gamma_\infty = 0.5 \) except for the largest \( C_v \) run which has 12\% error. These results were obtained by fitting the mixing length growth over the range \( 7 \leq L \). Thus it appears that, for \( \beta = \infty \) and \( C_v < 0.96 \), the asymptotic growth rate is achieved within a distance of 7 mesh blocks.

**Table 2.** Mixing length exponent \( \gamma \) for \( \beta = \infty \) for four sets of runs with different coefficients of variation \( C_v \).

In Fig. 3 we plot \( a_\infty \) versus \( C_v \) for \( \beta = \infty \) as measured from the runs in Table 2. The variation does indeed appear linear, except for the largest \( C_v \) used, suggesting that weak field theory is no longer appropriate above this value.

**Fig. 3.** The mixing length asymptotic scale factor \( a_\infty \) versus \( C_v \) for the field \( \beta = \infty \).

To check numerical convergence of the fluid flow/mixing length computations,
a mesh refinement study was performed for the $\beta = \infty$, $C_v = 0.38$ permeability field data (line 4 of Table 1). Fig. 4a shows the mixing length growth obtained on a fluid grid (32 x 256) equal in size to the grid used to generate the permeability data. The mixing length growth obtained on a fluid grid of (64 x 512) (at the limits of the computational memory available) is displayed in Fig. 4b. The computed mixing length growth exponents, $\gamma$, agree with each other and with the asymptotic theory to within 2%.

Fig. 4. $ln(t)$ versus $ln(L)$ for $\beta = \infty$, $C_v = 0.38$ computations obtained on flow grids of a) 32 x 256 b) 64 x 512.

The $\beta = \infty$ results above indicate behavior consistent with pure fractal, Fickian behavior (with perhaps very rapidly decaying transients), at least in the small heterogeneity strength regime $C_v < 0.9$.

Next we describe numerical simulations for two non-Fickian diffusion processes. We begin with the consideration of transient anomalous diffusion. Theoretically [Z], fluid flow in fractal permeability fields with $\beta = 1$ should exhibit the slowest decay to asymptotic growth rate. Thus, of our three sets of runs, $\beta = 1.25$ should exhibit the most noticeable transient effects.

Figure 5a plots mixing length growth versus distance for the $\beta = 1.25$ case listed in Table 1. Superimposed is a least-squares straight line fit of slope 0.57, revealing the curvature of the data. In Fig. 5b, the $ln(r)$ axis is broken into thirds, each third fitted by a least-squares straight line. The slopes of these successive data fits reveal the transient decay towards the predicted asymptotic value of
$\gamma_\infty = 0.5 \ln |Z|$, a correlation of the form

$$<\delta v(0)\delta v(r)> = \frac{a}{(b + r)^\beta}$$

is chosen to provide a simple interpolation from short distance $\gamma(0) = 1$ behavior, to long distance $\gamma_\infty$ asymptotics for the mixing length. Velocity correlations of the form (10) result, to leading order, from rock correlations of the same form, as can be seen from (5). (The second term on the right hand side has a much more rapid decay than the first for $\beta < 2$.) Using (10), with $a$, $b$ as free parameters and $\beta_\infty = \beta(\infty) = 1.25$, we fit the computed diffusion length data. The fitted profile, compared with the computational data, is displayed in Fig. 6. The best fit parameters obtained are $a = 3.8 \times 10^{-5}$, $b = 0.27$.

---

**Fig. 5.** Behavior of $\ln(l)$ versus $\ln(L)$ for the $\beta = 1.25$ runs. (a) Straight line fit shows non-fractal effects at short distance. (b) Straight line fits to successive thirds of the data exhibit evidence of a transient approach to a limit of 0.5.

Finally, we discuss anomalous diffusion generated by slowly decaying fractal rock statistics. Figure 7 displays the mixing length growth for the $\beta = 0.5$ case listed in Table 1. Omitting the range $L < 7$, we find the data well fit by a straight
Fig. 6. Two parameter, multi-fractal fit based on (10) to the mixing length growth data for the $\beta = 1.25$ runs. Best fit parameters are $a = 3.8 \cdot 10^{-5}$, $b = 0.27$.

Fig. 7. $\ln(t)$ versus $\ln(L)$ for the $\beta = 0.5$ case.

line having slope $\gamma = 0.82$, in 9% agreement with the theoretical asymptotic limit of 0.75. This behavior is consistent with non-Fickian fractal response. Studies such as performed in Figs. 5 and 6 for the $\beta = 0.5$ data do not indicate long time transient decay similar to that seen in the $\beta = 1.25$ data.
4 Discussion and Conclusions

We have described a new theory which provides quantitative predictions for the growth rate of the mixing length in the case of multi-fractal heterogeneity functions. This theory is supported by our present computational studies, except for very large permeability heterogeneity strength. We identify three distinct growth types dependent on the fractal exponent for the correlation function of the log-normal rock permeability: Fickian (normal) steady state growth, non-Fickian transient behavior slowly decaying to asymptotic (either Fickian or non-Fickian) rate, and non-Fickian steady state growth.

References


This article was processed using the LaTeX macro package with ICM style