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Some questions regarding the understanding and prediction of turbulent flow

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Abstract. Recent progress and remaining challenges related to the understanding and prediction of turbulent flow are described. This is done by explaining the need for the development and basic features of unified turbulence models which integrate, recover and extend well known existing computational methods for turbulent flow calculations. The availability of unified turbulence models enables a significant progress with regard to both our understanding of basic mechanisms of turbulence and efficient and accurate numerical predictions of turbulent flows. However, the development of a complete mathematical theory of the stochastic integro-differential equations involved still represents a significant challenge.
Keywords. Turbulence control, turbulence theory, unified turbulence modeling, stochastic methods, deterministic methods

1 Introduction

The ability to manipulate a flow field actively or passively [1-4] to realize a desired change is of immense technological importance. The potential benefits of developing efficient flow-control systems range from saving billions of dollars in annual fuel costs for land, air, and sea vehicles to achieving economically and environmentally more competitive industrial processes involving fluid flows. Internal combustion engine, energy generation by thermal power, passenger and cargo service by automobiles, airplanes, or ships, manufacturing by using fluidization technology in steel or chemical industry: all of these technologies are related to turbulent flow control.

Computational science represents a unique tool for the realization of flow control: numerical predictions of the behavior of turbulent flows enable detailed studies of the efficiency of flow control measures. However, the Navier-Stokes equations cannot be used directly for such numerical predictions: the computational costs of such direct numerical simulation (DNS) do not allow applications to most engineering and environmental flows [5-7]. Hence, one has to develop equations for averaged flow variables to reduce the computational costs. The development of such equations requires the use of modeling assumptions regarding the characteristic properties of turbulent motions. Basic contributions to advance our understanding of turbulent motions were provided by Kolmogorov [8-10]. Unfortunately, there are still significant shortcomings of our insight into basic mechanisms of turbulence which imply corresponding shortcomings of computational methods for turbulent flow predictions.

The objective of this paper is to describe these open questions and to illustrate a novel computational approach to overcome the problems. The paper is organized in the following way. Sections 2 and 3 describe open questions related to our understanding and the prediction of turbulent flow, respectively. A new computational methodology to address these questions will be described in section 4. These developments, the use of these methods for the solution of currently unsolved questions and remaining challenges will be summarized in section 5.

2 Understanding of turbulent flow

Our current understanding of basic mechanisms of turbulent flows (which represents the basis for the development of numerical methods for turbulent flow computations) is dominated by the ideas of Kolmogorov known as the K41 model [8-10]. Basic features of Kolmogorov's ideas and related unsolved questions will be described in sections 2.1 and 2.2 with regard to large-scale and small-scale turbulent motions, respectively.

2.1 Large-scale turbulent motions

Insight into the dynamics of large-scale turbulent motions is relevant to the development of ensemble averaged equations for turbulence (see section 3.1). First of all, one needs knowledge about the intensity and time or length scale of turbulent fluctuations. The intensity of turbulent fluctuations can be assessed by means of a transport equation for the turbulent kinetic energy k. This equation is, basically, determined by the equations for turbulent velocity fluctuations which are implied by the Navier-Stokes equations. The combination of this equation for k with a scale-determining equation represents a much bigger problem. There is no unique choice regarding the variable that has to be considered: one can use the characteristic time scale τ of turbulent motions, or the characteristic turbulence frequency $\omega = 1/\tau$, or the characteristic length scale $L = k^{1/2}\tau$ of large-scale turbulent eddies, or the dissipation rate $\epsilon = k/\tau$ of turbulent kinetic energy k [11-12]. Without providing any reasoning Kolmogorov suggested the consideration of ω [10]. Indeed, this choice appears to be the most appropriate one (at least for wall-bounded turbulent flows [12-13]). Changes of ω are governed by the transport equation [14]

$$\frac{\partial \omega}{\partial t} + \overline{U}_k \frac{\partial \omega}{\partial x_k} + \frac{1}{\langle \rho \rangle} \frac{\partial \langle \rho \rangle F_k}{\partial x_k} = -S_\omega \omega^2 \qquad (1)$$

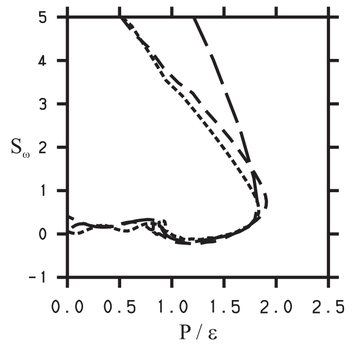

Figure 1: DNS data for the standardized source rate S_ω in equation (1) against the production-to-dissipation ratio P/ϵ of turbulent kinetic energy (see reference [13]).

Here, $\overline{U}_k$ is the averaged velocity (k = 1, 2, 3), and F_k is the turbulent frequency flux. The sum convention is applied throughout this paper. The turbulent frequency flux can be treated as the corresponding flux in the k-equation. Hence, the problem related to the closure of equation (1) is given by the need to specify the most important term in this equation: the standardized source rate S_ω. Unfortunately, the basis for explaining the structure of S_ω is weak. To assess S_ω one has to determine the rate at which energy is transferred through the spectrum, this means S_ω is the result of both production of turbulence at large scales and dissipation at small scales. Kolmogorov's notion was that ω is associated with the smallest scales of turbulence, and thus has no direct interaction with the mean motion. Correspondingly, he concluded that S_ω should be independent of the production of turbulence and approximated by a constant [10]. Nevertheless, in most of the applications S_ω is considered as a linear function of the production-to-dissipation ratio P/ϵ of turbulent kinetic energy: $S_\omega = \alpha_2 - \alpha_1 P/\epsilon$ [12]. Fig. 1 demonstrates the suitability of these assumptions on S_ω by adopting DNS data of compressible turbulent channel flows [13]. Fig. 1 shows that S_ω cannot be seen to be constant, and it supports the view that S_ω varies linearly with P/ϵ only for values $S_\omega > 0.5$ (which corresponds to the near wall region). Thus, a correlation between S_ω and P/ϵ which applies to all the flow does not exist.

To get further insight into the structure of source rates in scale-determining equations, Yakhot and Orszag [15] and Yakhot et al. [16] addressed this question regarding the source rate S_ϵ in the equation for the dissipation rate ϵ by means of the renormalization group (RNG) theory. Smith and Reynolds [17] found some algebraic errors in the derivation of Yakhot and Orszag. They concluded that this RNG-based theory does not provide support for assuming that S_ϵ is a function of P/ϵ . To obtain better agreement with general practice of modeling S_ϵ, Yakhot and Smith [18] revised the RNG-theory by modifying several underlying assumptions. This led to the standard form of the dissipation equation plus an additional unknown source term. However, Smith and Woodruff state: "Even though they may be motivated physically or otherwise, it is evident that many steps of the renormalization-group scale-removal procedure as currently formulated are mathematically not rigorously justified" [19].

In summary, the modeling of the standardized source rate S_ω in the turbulence frequency equation (1) and the corresponding modeling of S_ϵ in the dissipation rate equation still represent open questions. It is worth noting that it is possible to develop accurate S_ω models for some classes of flows, but the range of applicability of such models is limited (one has to use, for example, different models for incompressible and compressible flows) [13].

2.2 Small-scale turbulent motions

Insight into the dynamics of small-scale turbulent motions is relevant to the development of spatially filtered equations for turbulence (see section 3.2). The K41 hypotheses assume that, in intense turbulence and well away from any boundaries or singularities, the statistics of turbulent flow should be universal at length and time scales that are small compared with the injection of energy into the flow. If the small-scale statistics are to be universal, they must be independent of the large-scale flow structure. In particular, the K41 hypotheses predict that at small scales the turbulence should forget any preferred directions of the large-scale flow and that the small-scale fluctuations should be statistically homogeneous and isotropic [8-9].

However, the validity of Kolmogorov's assumption of the local isotropy of small-scale turbulent motions at high Reynolds numbers is still unclear [20-21]. Some experiments support Kolmogorov's hypothesis of isotropy [22], but other experiments clearly find anisotropic small scales [23-26]. Brasseur and Yeung [27] and Brasseur [28] analyzed the Navier-Stokes equations in Fourier space and came to the following conclusion: in a turbulent flow the coupling between the large and the small scales persists and is dynamically significant in the infinite-Reynolds number limit. Anisotropy of the small-scales can be induced, therefore, by the anisotropy of the large scales. Thus, a finite level of small-scale anisotropy may always exist if the large scales are anisotropic. Kerstein proposed a simple, empirically motivated model to explain the transfer of imposed large scale anisotropy to small scales in high-Reynolds-number turbulence [29]. However, a theory of how non-universal large-scale turbulent motions affect small-scale turbulence is still missing.

A related question concerns the relevance of small-scale anisotropy. By measuring the Lagrangian second-order structure function and velocity spectrum, Ouellette et al. [30] found that axisymmetry of a large-scale flow is also present in the small-scale fluctuations - which disagrees with Kolmogorov's hypothesis of local isotropy. In particular, the Kolmogorov constant C_0 in the Lagrangian second-order structure function was found to be $C_0 = C_0(\infty)/[1 + 365 Re_\lambda^{-1.64}]$, where Re_λ is the Taylor-scale Reynolds number [30]. The asymptotic values of C_0 at infinitely high Re are $C_0(\infty) = 6.2 \pm 0.3$ for the radial structure function and $C_0(\infty) = 5.0 \pm 0.4$ for the axial structure function. However, the parameter that controls small-scale turbulence dynamics is not C_0 but $c_0 = C_0/[C_0 + 2/3]$, see section 4, and the asymptotic values $c_0(\infty) = 0.9$ and $c_0(\infty) = 0.88$ for the radial and axial structure function, respectively, are almost the same.

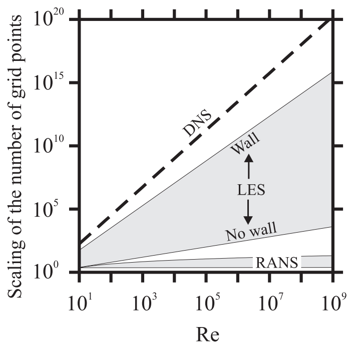

Figure 2: An illustration of the scaling of the number of grid points related to different methodologies in dependence on the Reynolds number Re. The use of DNS is characterized by a $Re^{9/4}$ scaling [41-42]. In regions away from a solid wall the number of grid points required for LES scales with $Re^{0.4}$, but the resolution of near-wall motions requires a number of grid points that increases proportional to $Re^{1.76}$ [43]. In contrast, the RANS grid requirements are proportional to ln(Re) [7, 44]. Away from a solid wall the RANS grid is independent of Re.

3 Prediction of turbulent flow

The questions related to the understanding of turbulent flow described in section 2 imply corresponding problems of numerical prediction methods. The implications of these questions will be described with regard to large-scale and small-scale turbulence models in sections 3.1 and 3.2, respectively.

3.1 Large-scale turbulence models

The huge computational costs of DNS imply the need to develop equations for averaged flow variables. The simplest way to obtain such equations is given by the use of an ensemble average to filter the Navier-Stokes equations, which results in deterministic Reynolds-averaged Navier-Stokes (RANS) [7, 12, 14, 31] or stochastic probability density function (PDF) methods [7, 14, 32-33]. The appearance of unknown correlations of turbulent fluctuations requires closure assumptions to close such RANS and PDF equations. Due to the reference to ensemble averages, such correlations involve turbulent fluctuations at all scales (an ensemble average may be seen as a filtering in space where the filter size is large compared to the characteristic size of large-scale turbulent eddies [14]). In particular, these correlations are dominated by large-scale fluctuations. Therefore, the development of RANS and PDF methods means, essentially, to model the non-universal mechanism of large-scale turbulent motions.

As pointed out in section 2.1, the essential problem of this approach is given by the need to explain the characteristic time or length scales of large-scale turbulent eddies. The generality of concepts applied currently to provide a scale-determining equation is certainly questionable [13]. Thus, RANS and PDF methods provide predictions that require evidence with regard to each flow considered - a problem which cannot be solved in general.

3.2 Small-scale turbulence models

The basic reason for the problems of RANS and PDF methods described in section 3.1 is given by the reference to ensemble averages (which involves the need to explain the dynamics of large-scale turbulent motions). An obvious way to overcome these problems is given by the use of a filtering in space such that modeling assumptions are restricted to small-scale turbulent motions. The latter approach leads to deterministic large eddy simulation (LES) [7, 34-36] or stochastic filter density function (FDF) methods [14, 37-40]. Unfortunately, the use of these methods may require infeasible computational costs. FDF simulations are six times less expensive than DNS, but they may require 15-30 times more effort than LES methods [37]. Similarly, the use of LES will be too expensive for many applications: Fig. 2 shows that the number of grid points required for wall-bounded flow simulations is nearly the same as that required for performing DNS [41-42]. Consequently, such simulations become infeasible for high-Reynolds number flows such as occur in aeronautical and meteorological applications.

The question regarding the relevance of accounting for effects of non-universal large-scale motions on small-scale turbulence is very important regarding the feasibility of LES and FDF calculations. For example, the consideration of an anisotropic relaxation of velocity fluctuations adds a lot of complexity to both modeling and simulations (see the discussion related to G^d_{ij} in section 4.2). Unfortunately, the discussion in section 2.2 shows that a clear answer to this question is currently unavailable.

4 A new computational approach

It will be shown in section 4.1 that the problems described in section 3 can be solved by unifying existing turbulence models. The question of how such unified models can be developed will be addressed in sections 4.2 and 4.3.

4.1 The need for unified models

Due to the high computational requirements related to FDF and LES methods, at least the simulation of high-Reynolds number wall-bounded flows requires, therefore, combinations of FDF and PDF, or LES and RANS methods. The use of consistent combinations appears to be helpful to perform accurate computations (see, for example, the discussion of the mean velocity mismatch by Hamba [45]). Thus, there is a need for the development of unified turbulence models that may be used depending on the resolution as FDF or PDF, LES or RANS methods. From a more general point of view, the development of such methods appears to be helpful to improve the efficiency and accuracy of turbulence simulations: accurate FDF and LES calculations could be applied to assess and improve the performance of more efficient PDF and RANS calculations. Also from a theoretical point of view, the development of unified turbulence models is relevant: a common modeling strategy may well contribute to model developments.

However, the construction of unified turbulence models turned out to represent a non-trivial problem. Previously presented suggestions addressed this problem by considering the question of how it is possible to find a generalized stress tensor that recovers the stress tensors in the RANS and LES limits [14, 35, 46-48]. The stress tensor is the result of a variety of physical processes. Thus, it is impossible to find support for the construction of a generalized stress tensor by taking reference to a simple physical explanation of scale effects. The only way to develop generalized stress models is then given by the application of heuristic interpolation procedures between RANS and LES limits. Many suggestions were presented for that, and the assessment of differences between such suggestions turned out to be a complicated problem [47].

4.2 Unified stochastic models

As an alternative to the deterministic approaches applied previously, the development of unified turbulence models is addressed at the more general level of stochastic equations [33]. The generalized Langevin model represents an appropriate basis for the modeling of turbulent velocities,

$$\frac{dx_i^*}{dt} = U_i^*, \tag{2}$$

$$\begin{aligned} \frac{dU_i^*}{dt} &= \frac{2}{\langle\rho\rangle_{\mathrm{G}}}\frac{\partial\langle\rho\rangle_{\mathrm{G}}\widetilde{\nu}\widetilde{S}_{ik}^d}{\partial x_k} - \frac{1}{\langle\rho\rangle_{\mathrm{G}}}\frac{\partial\langle p\rangle_{\mathrm{G}}}{\partial x_i} \\ &\quad -\frac{1}{\tau_L}\left[\delta_{ik} - \tau_L G_{ik}^d\right]\left(U_k^* - \widetilde{U}_k\right) + \sqrt{\frac{4c_0k}{3\tau_L}}\frac{dW_i}{dt}. \end{aligned} \tag{3}$$

Here, x_i^* and U_i^* represent the ith components of a fluid particle position and velocity. The inclusion of the spatial transport terms (the first two terms on the right-hand side of (3)) assures the correct transport of filtered velocities in inhomogeneous flows. $\langle\rho\rangle_{\mathrm{G}}$, $\langle p\rangle_{\mathrm{G}}$, $\widetilde{\nu}$, and $\widetilde{S}_{ij}^d$ denote the filtered fluid mass density, pressure, viscosity and deviatoric rate-of-strain tensor, respectively (ν and S_{ij}^d are assumed to be uncorrelated for simplicity). The generation of fluctuations is described by the noise term (the last term in (3)) which is determined by the properties of dW_i/dt. The latter is a Gaussian process with vanishing means, $\langle dW_i/dt\rangle = 0$, and uncorrelated values at different times, $\langle dW_i/dt(t)dW_j/dt(t')\rangle = \delta_{ij}\delta(t-t')$. Here, δ_{kl} is the Kronecker delta and $\delta(t-t^{'})$ is the delta function. k represents the residual turbulent kinetic energy defined by $k = D_{kk}/2$, where D_{ij} represents the subgrid-scale (SGS) stress tensor (the variance of the FDF). The noise strength is controlled by the nondimensional parameter c_0. An analysis of implications of (3) reveals that $c_0 = 19/27 \approx 0.7$ [39]. DNS results [37] support this finding very well, and investigations of effects of c_0 variations on simulation results [37-38] also confirm the suitability of $c_0 \approx 0.7$. A detailed discussion of variations of this parameter may be found elsewhere [33]. The relaxation term (the third term on the right-hand side of (3)) models a relaxation towards the filtered velocity $\widetilde{U}_k$ with a characteristic relaxation time scale τ_L. The consideration of G_{ik}^d enables an accurate representation of the anisotropy of stresses. The closure of G_{ik}^d in terms of turbulence statistics is addressed elsewhere [33].

The filtered variables involved in (2)-(3) are defined as spatially filtered variables. The properties of the stochastic model (2)-(3) depend essentially on the choice of the filter function that is controlled, basically, by the filter width Δ. In particular, the properties of (2)-(3) are determined by the ratio of Δ to several characteristic length scales. A first case (the small-Δ limit) is given by adopting a small filter width $\Delta << L$, where L is the characteristic length scale of large-scale motions. FDF / LES calculations are performed by adopting $\eta << \Delta << L$ (η is the Kolmogorov length scale), and DNS is performed by adopting $\Delta << \eta$ [33]. A second case (the large-Δ limit) is given by choosing Δ very large: $L << \Delta << L_0$, where L_0 characterizes the length scale of largest eddies that are injected into the flow. To prove the suitability of the stochastic model (2)-(3) as a unified turbulence model, let us compare the small-Δ and large-Δ limits of the model (2)-(3) with FDF and PDF models applied previously.

Regarding the small-Δ limit of (2)-(3) one observes only one difference to FDF models applied previously: existing FDF models are generalized by involving the deviatoric frequency G_{ik}^d. This consideration of G_{ik}^d is an option which assures a common structure of FDF and PDF models and an accurate representation of the anisotropy of stresses, but this approach may be related to some disadvantages [33]. Another option is given by the neglect of G_{ik}^d in (2)-(3). The suitability of the resulting FDF model was proved in a variety of investigations and applications [37-38, 40], and the resulting PDF model represents a reasonable model [14, 49-50] as long as the gradient Mach number is smaller than unity [51].

Regarding the large-Δ limit of the stochastic model (2)-(3) one observes that these equations recover the generalized Langevin PDF model with one exception: spatially filtered variables are applied in (2)-(3) whereas ensemble averages are used in the generalized Langevin PDF model. An ensemble average is defined as mean over all possible values of a variable considered at any position and time. The use of a sufficiently large filter width $\Delta >> L$ is then an obvious requirement to involve values of all the energy spectrum. Do such spatially averaged variables represent ensemble means? The latter is strictly only the case if turbulence statistics do not change in space, this means for statistically homogeneous flows [52]. With regard to inhomogeneous flows one calculates variables in this way which are both ensemble-averaged (in the sense that all possible values of a variable considered are involved) and smoothed in space, this means one calculates variables which are somewhat coarser than strict ensemble means. The use of such smoothed ensemble means represents a valid concept provided the smoothing in space allows an appropriate characterization of spatial variations of the flow considered. Exactly this type of averaging is applied to solve PDF transport equations numerically (in stochastic particle methods one calculates ensemble averages as means over an ensemble of particles inside a box considered [7]). Thus, the large-Δ limit of the equations (2)-(3) represents a PDF model.

The stochastic model (2)-(3) is unclosed as long as the time scale τ_L of turbulent motions is not defined. In particular, τ_L should recover the corresponding PDF and FDF limits of τ_L. We assume that $\tau_L = \ell k^{-1/2}$. The characteristic length scale ℓ of turbulent fluctuations will be defined by $\ell = \ell_* T_\lambda L$, where $L = k^{1/2}\tau$. Hence,

$$\tau_L = \ell_* T_\lambda \tau, \tag{4}$$

where $\ell_* = 1/3$. The transfer function T_λ is given by [33]

$$T_\lambda = \int_0^{\Delta/L} \theta_\lambda(1-y)dy. \tag{5}$$

The transfer function T_λ depends on the specification of the distribution function $\theta_\lambda(1-y)$. This distribution function represents a smooth generalization of the theta function depending on any parameter λ. For $\lambda \to 0$, we find $\theta_\lambda(1-y) = \theta(1-y)$, where $\theta(1-y)$ refers to the theta function. An analysis of the effect of different choices of nonzero λ values reveals a relatively low relevance of nonzero λ values [33]. Thus, it appears to be well justified to consider T_λ to be independent of λ. The corresponding limit $T = \lim_{\lambda\to 0} T_\lambda$ is given by

$$T = \int_0^{\Delta/L} \theta(1-y)dy = \begin{cases} \Delta/L: & \Delta/L < 1 \\ 1: & \Delta/L \geq 1 \end{cases} \tag{6}$$

Expression (4) combined with $T_\lambda = T$ generalizes the characteristic time scales used in large-scale and small-scale turbulence models [33].

4.3 Unified deterministic models

Deterministic unified turbulence models can be obtained by rewriting the stochastic model (2)-(3) into a Fokker-Planck equation for the FDF, multiplying this equation with the corresponding variables and integrating over the velocity sample space. In this way, one obtains the following equations for the filtered mass density $\langle\rho\rangle_{\mathrm{G}}$, velocity $\widetilde{U}_i$ and SGS stress D_{ij} [33],

$$\frac{\widetilde{D}\langle\rho\rangle_{\mathrm{G}}}{\widetilde{D}t} = -\langle\rho\rangle_{\mathrm{G}}\frac{\partial\widetilde{U}_k}{\partial x_k}, \tag{7}$$

$$\frac{\widetilde{D}\widetilde{U}_i}{\widetilde{D}t} + \frac{1}{\langle\rho\rangle_{\mathrm{G}}}\frac{\partial\langle\rho\rangle_{\mathrm{G}}D_{ik}}{\partial x_k} = \frac{2}{\langle\rho\rangle_{\mathrm{G}}}\frac{\partial\langle\rho\rangle_{\mathrm{G}}\widetilde{\nu}\widetilde{S}^d_{ik}}{\partial x_k} - \frac{1}{\langle\rho\rangle_{\mathrm{G}}}\frac{\partial\langle p\rangle_{\mathrm{G}}}{\partial x_i}, \tag{8}$$

$$\frac{\widetilde{D}D_{ij}}{\widetilde{D}t} + \frac{1}{\langle\rho\rangle_{\mathrm{G}}}\frac{\partial\langle\rho\rangle_{\mathrm{G}}T^D_{kij}}{\partial x_k} + D_{ik}\frac{\partial\widetilde{U}_j}{\partial x_k} + D_{jk}\frac{\partial\widetilde{U}_i}{\partial x_k} = -\frac{2}{\tau_L}\left(D_{ij} - \frac{2}{3}c_0 k\delta_{ij}\right). \tag{9}$$

Here, $\widetilde{D}/\widetilde{D}t = \partial/\partial t + \widetilde{U}_k\partial/\partial x_k$ denotes the filtered Lagrangian time derivative, and T^D_{kij} is the triple correlation tensor of SGS velocity fluctuations. An analysis of equation (9) reveals that this equation recovers and extends well known stress models [33]. Corresponding dynamic formulations of these stress models can also be used, which enables the application of different parameter values for various flow regions [14, 33, 39, 53].

5 Solved and unsolved questions

The following conclusions can be drawn. The development of numerical prediction methods enables detailed studies of the efficiency of flow control measures. Unfortunately, our limited understanding of basic mechanisms of turbulent flows implies significant shortcomings of existing computational methods. The most promising way to overcome these problems is given by the development of unified turbulence models. It was shown that such unified stochastic models recover well known FDF and PDF models. The implied unified deterministic models recover and extend well known LES and RANS models. One advantage of unified turbulence models is given by the fact that the problems of existing computational methods are solved in this way: accurate FDF and LES calculations can be applied to assess and improve the performance of more efficient PDF and RANS calculations. Another advantage of unified turbulence models is given by the fact that the open questions related to our understanding of turbulence can be addressed: unified turbulence models can be used, e.g., to study the question of how non-universal large-scale turbulent motions affect small-scale turbulence, and they can be used to calculate the length and time scales of large-scale turbulent motions.

However, one has to note that the mathematical theory of the stochastic integro-differential equations considered is not yet fully developed: for example, general conditions for the existence and uniqueness of solutions of these equations are still unknown. Another important question concerns the most appropriate strategy to solve these equations numerically: one has the choice between several options. Additional questions are related to the most appropriate way to study these equations: they can be formulated in correspondence to both ordinary stochastic differential equations and partial stochastic differential equations [54], and the influence of such formulations on the numerical solution of these equations is relatively unknown until now. Such research represents, therefore, a significant challenge.

6 References

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