On the Kolmogorov constant in stochastic turbulence models

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The Kolmogorov constant is fundamental in stochastic models of turbulence. To explain the reasons for observed variations of this quantity, it is calculated for two flows by various methods and data. Velocity fluctuations are considered as the sum of contributions due to anisotropy, acceleration fluctuations and stochastic forcing that is controlled by the Kolmogorov constant. It is shown that the effects of anisotropy and acceleration fluctuations are responsible for significant variations of the Kolmogorov constant. It is found near 2 for flows where anisotropy and acceleration fluctuations contribute to the energy budget, and near 6 if such contributions disappear. [DOI: 10.1063/1.1514217]

Most of the stochastic Lagrangian models for the simulation of turbulent flows involve specific formulations of the generalized Langevin model for the simulation of velocities $U_i^*(i=1,3)$,[1]

$$\frac{d}{dt}U_i^* = -\rho^{-1}\frac{\partial p}{\partial x_i} + G_{ij}(U_j^* - \langle U_j\rangle) + \sqrt{C_0\epsilon}\frac{dW_i}{dt}. \tag{1}$$

The first term on the right-hand side determines the dynamics of mean velocities. p and ρ are the ensemble-averaged pressure and mass density. The last term models the generation of fluctuations. dW_i/dt is the ith component of a vectorial Gaussian process with vanishing means, $\langle dW_i/dt\rangle = 0$, and uncorrelated values at different times, $\langle dW_i/dt(t)\cdot dW_j/dt'(t')\rangle = \delta_{ij}\delta(t-t')$. δ_{ij} denotes the Kronecker delta, $\delta(t-t')$ the delta function and $\langle\cdots\rangle$ refers to the ensemble average. The intensity of noise generation is proportional to the mean dissipation rate ϵ of turbulent kinetic energy. The second term on the right-hand side of (1) models the relaxation of velocity fluctuations. Repeated indices imply summation, and $\langle U_j\rangle$ is the jth component of the mean Eulerian velocity. The closure of (1) requires the specification of G_{ij} and the constant C_0. Usually, the coefficients G_{ij} are chosen as algebraic functions of velocity gradients and variances $\langle u_i u_j\rangle$ (u_i denotes an Eulerian velocity fluctuation). Such expressions for G_{ij} may be obtained as solutions of the algebraic equations

$$C_0\epsilon\delta_{ij} = \left(\frac{\partial\langle U_i\rangle}{\partial x_k} - G_{ik}\right)\langle u_k u_j\rangle + \left(\frac{\partial\langle U_j\rangle}{\partial x_k} - G_{jk}\right)\langle u_k u_i\rangle, \tag{2}$$

which are found by neglecting the gradients of variances and triple correlations in the variance equations given by (1). By adopting (2) to provide G_{ij}, the remaining task is the determination of C_0.

One may expect that C_0 grows with the Reynolds number and approaches to an asymptotic value $C_0(\infty)$ that is called the Kolmogorov constant. This was confirmed by Sawford. He showed that $C_0(\infty)=7$ for homogeneous isotropic stationary turbulence (HIST).[2] Recently, Sawford and Yeung derived a revised value $C_0(\infty)=6$ on the basis of new direct numerical simulation (DNS) data,[2,3] which agrees with conclusions of Pope.[4] $C_0(\infty)$ should be a universal constant according to Kolmogorov's theory, i.e., the same for each flow. However, many investigations did not confirm this assumption. In other flows than HIST (for decaying turbulence, evolving scalar fields and the atmospheric boundary layer), one found smaller values $1\leqslant C_0(\infty)\leqslant 3$.[5–10] The reasons for the obtained variation of $C_0(\infty)$ estimates between 1 and 7 were not explained until now.

Obviously, a better knowledge of the reasons for the variations of $C_0(\infty)$ is relevant to stochastic simulations of turbulence. To address this question, it is helpful to write (1) as

$$U_i^* - \langle U_i^*\rangle = u_i^{\mathrm{an}} + u_i^{\mathrm{ac}} + u_i^{\mathrm{st}}. \tag{3a}$$

The contributions due to anisotropy, acceleration fluctuations, and stochastic forcing are given by

$$\begin{aligned} u_i^{\mathrm{an}} &= [\delta_{ij} - G^{-1}G_{ij}](U_j^* - \langle U_j\rangle), \\ u_i^{\mathrm{ac}} &= G^{-1}\left(\frac{dU_i^*}{dt} - \left\langle\frac{dU_i^*}{dt}\right\rangle\right), \\ u_i^{\mathrm{st}} &= G^{-1}\sqrt{C_0\epsilon}\frac{dW_i}{dt}. \end{aligned} \tag{3b}$$

Equations (3a) and (3b) are obtained by introducing the fluctuation of dU_i^*/dt and splitting G_{ij} into its isotropic ($G = G_{nn}/3$) and deviatoric part. This writing of (1) suggests to relate the appearance of various $C_0(\infty)$ values to the relevance of u_i^{an} and u_i^{ac}: high values of $C_0(\infty)$ are needed as a compensation for the disappearance of u_i^{an} and u_i^{ac} contributions to the variance (energy) budget in some flows or their neglect in flow simulations. Facts to support this view will be presented here.

C_0 will be calculated for an equilibrium turbulent boundary layer (ETBL) and HIST by adopting various models as pointed out in Table I. The calculation of $C_0^{(\mathrm{I})}$ is required to obtain an accurate asymptotic value $C_0(\infty)$ of C_0 for a flow that constitutes a cornerstone for the calculation of

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TABLE I. Calculations of C_0 to be performed for the equilibrium turbulent boundary layer (ETBL) and homogeneous isotropic stationary turbulence (HIST) by means of various models.

Model	ETBL	HIST
Eq. (3a) with $u_i^{\mathrm{an}}\neq 0$; $u_i^{\mathrm{ac}}\neq 0$	$C_0^{(\mathrm{I})}$	$(=C_0^{(\mathrm{IV})})$
Eq. (3a) with $u_i^{\mathrm{an}}=0$; $u_i^{\mathrm{ac}}\neq 0$	$C_0^{(\mathrm{II})}$	$(=C_0^{(\mathrm{IV})})$
Eq. (3a) with $u_i^{\mathrm{an}}=u_i^{\mathrm{ac}}=0$	$C_0^{(\mathrm{III})}$	$C_0^{(\mathrm{IV})}$

wall-bounded turbulent flows of engineering and environmental interest. This is needed to assess the relevance of Reynolds number effects on C_0, i.e., to clarify the question whether previously obtained low $C_0(\infty)$ estimates are the result of considering the flows at too small Reynolds numbers or not. $C_0^{(\mathrm{II})}$ and $C_0^{(\mathrm{III})}$ will be calculated for the same flow to see the effect of the neglect of u_i^{an} and u_i^{ac}. The consideration of these simpler models is relevant because they are used for most of the stochastic simulations of reacting flows.[1] $C_0^{(\mathrm{IV})}$ will be calculated for HIST for which contributions of u_i^{an} and u_i^{ac} to the variance budget disappear. Thus, the value of C_0 obtained for HIST has to be the same for the models considered here. Evidence for this will be provided by the comparison with the results of Sawford's acceleration model,[2] which is (for HIST) more complete than Eq. (3a). The purpose of calculating $C_0^{(\mathrm{IV})}$ is to assess the flow dependence of C_0. In addition to this, it is of interest to see if there is any difference between the disappearance of u_i^{an} and u_i^{ac} contributions to the variance budget in some flows and the neglect of such contributions in other flows where they are nonzero.

The calculation of $C_0^{(\mathrm{I})}$ requires the specification of G_{ij} in model (1). Very often, G_{ij} is taken according to the simplified Langevin model (SLM), which is used in general to perform reacting flow calculations. However, this implies the assumption of relations between $\langle u_1u_1\rangle$, $\langle u_2u_2\rangle$ and $\langle u_3u_3\rangle$ that are only satisfied approximately for the ETBL, see below. To overcome this shortcoming of the SLM, we will use (2) in combination with the known statistics of the ETBL to derive a more general parametrization of G_{ij}. For the flow considered, relation (2) provides four equations for 10 unknowns (the nine components of G_{ij} and C_0) and two consistency constraints that assure $\langle u_1u_3\rangle=\langle u_2u_3\rangle=0$ (the coordinate system is chosen such that x_1 is in the streamwise direction and x_2 is in the direction of the mean shear). One may distinguish two types of models in dependence on the choice of G_{ij}: models where C_0 does not affect the transport of turbulent kinetic energy [the variance equations (2)], and models where it controls this transport (e.g., the SLM). We are interested in the second model type because it is used in general for stochastic flow simulations. Such a model is obtained by generalizing the SLM through choosing G_{ij} to be diagonal ($G_{ij}=0$ for $i\neq j$), i.e., we assume that each velocity component interacts primarily with its mean, see (1). This assumption is in agreement with the two consistency constraints mentioned above. It reduces the number of unknowns to four, for which four equations are provided through (2). By adopting the relations $\langle u_1u_3\rangle=\langle u_2u_3\rangle=0$ and $\langle u_1u_2\rangle=-\epsilon S^{-1}$ for the variances and $\partial\langle U_i\rangle/\partial x_k$ $=S\delta_{i1}\delta_{k2}$ for the gradients, the equations for G_{11}, G_{22}, G_{33}, and C_0 read

$$-0.5C_0\epsilon=\epsilon+G_{11}\langle u_1u_1\rangle=G_{22}\langle u_2u_2\rangle=G_{33}\langle u_3u_3\rangle, \tag{4a}$$

$$S\langle u_2u_2\rangle=(G_{11}+G_{22})\langle u_1u_2\rangle. \tag{4b}$$

G_{11}, G_{22}, and G_{33} are found according to (4a), and C_0 can be calculated from (4b) as

$$C_0^{(\mathrm{I})}=2\{\langle u_1u_1\rangle\langle u_2u_2\rangle/\langle u_1u_2\rangle^2-1\}(1+\langle u_1u_1\rangle/\langle u_2u_2\rangle)^{-1}. \tag{5}$$

The model that follows from (1) in conjunction with (5), $G_{ij}=0$ for $i\neq j$ and (4a) for G_{11}, G_{22}, and G_{33} will be referred to as Langevin model (LM). It is worth emphasizing that the consideration of more general forms of G_{ij} (e.g., off-diagonal components[3] or nonlocal expressions[11]) is no requirement for the flow considered: the LM is capable of providing the variances in agreement with DNS and measurements. An essential feature of the approach applied here is given through the fact that all the model ingredients (G_{11}, G_{22}, G_{33} and C_0) are determined through the assumption that G_{ij} is diagonal. C_0 was calculated according to (5) by adopting the channel flow DNS data of Moser *et al.*[12] and measurements of Wei and Willmarth[13] and Antonia *et al.*[14] at $y^+=98$, see Pope (p. 283).[1] The results are shown in Fig. 1 in dependence on the Reynolds numbers Re (based on the bulk velocity) and Re_0 (based on the centerline velocity).[1] This was done by relating the friction Reynolds number by $\mathrm{Re}_\tau=0.09\,\mathrm{Re}^{0.88}$ with Re, see Pope, p. 279.[1] These results reveal that the influence of the Reynolds number on these predictions is negligible (there is no observable trend of these values with the Reynolds number) and that the asymptotic value $C_0(\infty)$ is significantly less than 6. In particular, the DNS data provide a mean of $C_0=2.1\pm 0.04$. This value agrees well with the results of measurements: both provide $C_0=2.3\pm 0.3$.

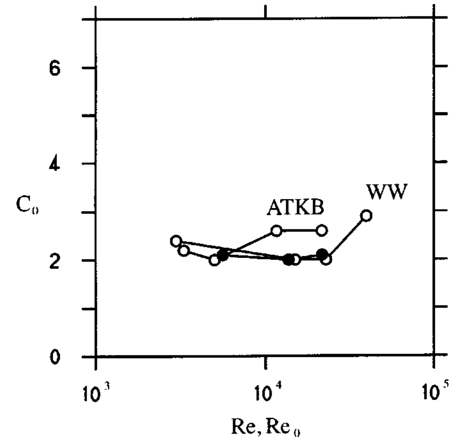

FIG. 1. The solid symbols show C_0 obtained for the LM in dependence on the Reynolds number Re according to the DNS data of Moser, Kim, and Mansour (Ref. 12). The open symbols represent C_0 in dependence on the Reynolds number Re_0 according to the measurements of Wei and Willmarth (referred to as WW) (Ref. 13) and Antonia *et al.* (referred to as ATKB) (Ref. 14).

The calculation of $C_0^{(\mathrm{II})}$ and $C_0^{(\mathrm{III})}$ requires the simplification of the LM. First, we assume $G_{11}=G_{22}=G_{33}=G$, i.e.,

TABLE II. Normalized DNS data (Ref. 12) needed to assess the approximations $\langle u_1u_1\rangle=\eta_1$ and $\langle u_2u_2\rangle=\langle u_3u_3\rangle=\eta_2$ ($\eta_1=\langle u_2u_2\rangle+2\langle u_1u_2\rangle^2/\langle u_2u_2\rangle$, and $\eta_2=0.5[\langle u_2u_2\rangle+\langle u_3u_3\rangle]$).

Re_τ	$\langle u_1u_1\rangle$	$\langle u_2u_2\rangle$	$\langle u_3u_3\rangle$	$\langle u_1u_2\rangle$	η_1	η_2
180	1.59	0.56	0.74	−0.42	1.19	0.65
395	2.73	0.95	1.42	−0.73	2.07	1.19
590	3.13	1.06	1.63	−0.80	2.27	1.35

we neglect u_i^{an} in (3a). The reduction of the numbers of coefficients implies two assumptions on the velocity field. The first assumption is given by $\langle u_2u_2\rangle=\langle u_3u_3\rangle$, see Eq. (4a). Then, Eq. (4a) provides two relations: one finds $G=-(3C_0+2)/(4\tau)$ and $C_0=[k/\langle u_2u_2\rangle-1.5]^{-1}$. The second assumption on the velocity field is found through Eq. (4b), which implies $\langle u_1u_1\rangle=\langle u_2u_2\rangle+2\langle u_1u_2\rangle^2/\langle u_2u_2\rangle$. By adopting these relations there are different ways to calculate C_0 by means of measured variances. The DNS data given in Table II reveal that the implied relations between the variances are only satisfied approximately (the deviations are 13–27%), which explains the appearance of various C_0 values in models of the ETBL.[6,8,10] $C_0=[k/\langle u_2u_2\rangle-1.5]^{-1}$ is used in conjunction with measurements of $k/\langle u_2u_2\rangle$, e.g., for the construction of second-order closure models for the atmospheric boundary layer, which corresponds with the use of small values of C_0.[7] These imbalances can be minimized by calculating C_0 from k combined with the normalization $\langle u_1u_2\rangle$, i.e., in terms of $C_\mu=\langle u_1u_2\rangle^2/k^2$.[6,8] The relations presented above provide

$$C_0^{(\mathrm{II})}=(2/3)(1+\sqrt{1-6C_\mu})(1-\sqrt{1-6C_\mu})^{-1}. \tag{6}$$

The model resulting from (1) combined with $G_{ij}=G\delta_{ij}$ and (6) is the SLM, which is often used for flow simulations.[1] $C_0^{(\mathrm{III})}$ is obtained by reducing (3a) to a diffusion model (DM). This is done by neglecting the turbulence in the streamwise direction[5] so that $dx_1^*/dt=\langle U_1\rangle$. By neglecting u_i^{an} and u_i^{ac} in (3a), the two-dimensional diffusion ($i=2,3$) is then described by

$$dx_i^*/dt=\partial\Gamma/\partial x_i+\sqrt{2\Gamma}\,dW_i/dt. \tag{7}$$

The coefficient $G=-C_0\epsilon/(2\eta_2)$ in u_i^{st} can be obtained by (2), where $\eta_2=0.5(\langle u_2u_2\rangle+\langle u_3u_3\rangle)$. This implies $\Gamma=2\eta_2^2/(C_0\epsilon)$ for the diffusion coefficient in (7). The systematic term $\partial\Gamma/\partial x_i$ may be found as consistency condition by considering the Fokker–Planck equation related to (7). It is worth noting that the model (7) is not specifically related to the LM or SLM: exactly the same model can be obtained by constructing a stochastic model under the constraint that the probability density function evolves towards a Gaussian shape (which differs from the LM through the appearance of nonlinear terms) and taking the asymptotic limit.[5,10] By adopting a flux-gradient relationship, C_0 may be calculated as explained by Rodean.[10] Flux-gradients experiments in the (high-Reynolds number) horizontally uniform neutral atmospheric surface layer show that $\Gamma=\langle u_1u_2\rangle^2/\epsilon$, which implies

$$C_0^{(\mathrm{III})}=2(\eta_2/\langle u_1u_2\rangle)^2. \tag{8}$$

C_0 was calculated according to the channel flow DNS data of Moser *et al.*[12] by means of (6) and (8). Experimental data could not be used for a comparison because $\langle u_3u_3\rangle$ was not measured.[13,14] The results are given in Fig. 2 in dependence on the friction Reynolds number Re_τ, which is convenient regarding the comparison with results obtained for $C_0^{(\mathrm{IV})}$, see the explanations in the next section. In agreement with the previous claim regarding the reasons of $C_0(\infty)$ variations, one finds that the use of simpler models (the SLM and DM) results in growing values of C_0.

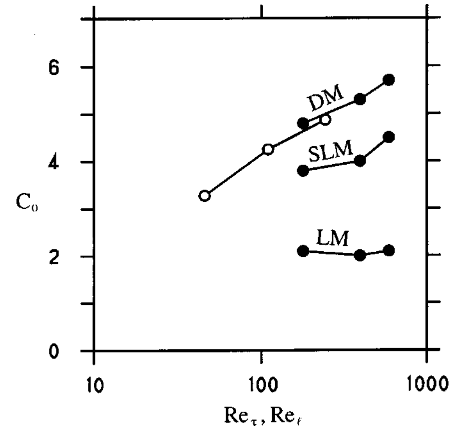

FIG. 2. The solid symbols are used to show the C_0 values of the LM, SLM, and DM according to the DNS data of Moser, Kim, and Mansour in dependence on Re_τ (Ref. 12). The open symbols present the temporal average values of C_0 according to (9) in dependence of Re_l, where the DNS data of Overholt and Pope were applied (Ref. 15).

$C_0^{(\mathrm{IV})}$ will be calculated by considering (7) with $i=1,3$ for the case of passive scalar mixing in HIST with imposed constant mean scalar gradient, which was studied by Overholt and Pope by means of DNS.[15] For the HIST considered, Γ is a constant equal to $\Gamma=8k^2/(9C_0\epsilon)$. The definition of Γ via the flux-gradient relationship $\langle u\phi\rangle=-\Gamma\partial\langle\Phi\rangle/\partial x$ ($\langle\Phi\rangle$ and ϕ are the mean and fluctuation of a passive scalar, and u and x refer to one component of u_i and x_i) implies then

$$C_0^{(\mathrm{IV})}=-(8k^2/9\epsilon\langle u\phi\rangle)(\partial\langle\Phi\rangle/\partial x). \tag{9}$$

The temporal average values of $C_0(=-2/\langle v^*\phi^*\rangle$ in the notation of Overholt and Pope) obtained for the stationary portion of each simulation are plotted in Fig. 2. This is done in dependence on the Reynolds number Re_l based on the turbulence intensity and integral length scale, which is advantageous regarding the comparison with the corresponding data obtained for the ETBL. The C_0 value at $\mathrm{Re}_l=1092$ (with a Taylor-scale Reynolds number $\mathrm{Re}_\lambda=185$) was not considered because it is strongly influenced by the forcing energy input.[15] The range $28\leqslant\mathrm{Re}_\lambda\leqslant 84$ considered in this way is larger than the range $\mathrm{Re}_\lambda<60$ considered by Weinman and Klimenko[9] and corresponds to the range considered by Sawford to calculate C_0 for the same flow.[2] We see that there is a very good agreement between the data provided by (7) for the ETBL and HIST. As pointed out above, the data obtained for $C_0^{(\mathrm{IV})}$ have to be consistent with the findings of Sawford[2] and Sawford and Yeung.[3] This is demonstrated in Fig. 3 where the $C_0^{(\mathrm{IV})}$ data are shown against Re_λ. Sawford's parametrization[2] for the Reynolds number dependence of C_0,

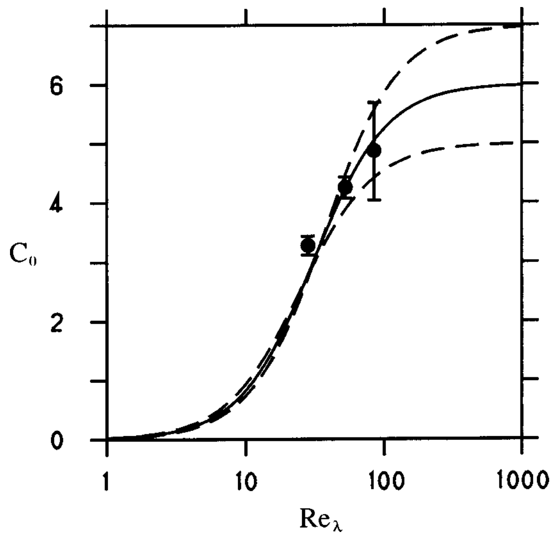

FIG. 3. The symbols show the C_0 values according to Overholt and Pope's DNS data (Ref. 15) as in Fig. 2 but now in dependence on Re_λ. The error bars denote the accuracy of these C_0 values. The solid line gives the prediction of Sawford's formula (10) for $C_0(\infty)=6$. The lower and upper dashed lines present the predictions of this formula for $C_0(\infty)=5$ and $C_0(\infty)=7$, respectively.

$$C_0=C_0(\infty)(1+7.5C_0^2(\infty)\mathrm{Re}_\lambda^{-1.64})^{-1}, \tag{10}$$

was used to explain the $C_0^{(\mathrm{IV})}$ variations. By adopting different values for $C_0(\infty)$, it is found that $C_0(\infty)=6$ represents the optimal choice. Consequently, the results obtained here are found to be fully consistent with the recent results of Sawford and Yeung[3] with reference to both the Reynolds number dependence of C_0 and the revised value $C_0(\infty)=6$. This is of relevance regarding the few data that are available for this flow and the questions related to their accuracy.[3]

The new findings about C_0 reported here may be summarized as follows. (i) Regarding the ETBL described by the LM there is a good agreement between the results obtained from DNS and measurements, $C_0=2.1$ and $C_0=2.3$, respectively. These C_0 values are found to be unaffected by the Reynolds number. Thus, previously obtained low $C_0(\infty)$ estimates (at least for the ETBL) are not the result of considering the flows at too small Reynolds numbers. The parametrization (10) does not have to be applicable to this case, but it is worth emphasizing that there is no contradiction between the Reynolds number independence reported here and (10): the second term in the denominator of (10) is for $C_0(\infty)=2.1$ about one order of magnitude smaller than for $C_0(\infty)=6$. (ii) The neglect of u_i^{an} and u_i^{ac} in (3a) (which results in the SLM and DM) provides higher values of C_0 that vary with the Reynolds number. The reason for this may be seen by means of (3a): higher C_0 values are needed as a compensation for the neglect of u_i^{an} and u_i^{ac}. This finding is relevant to applications because the SLM and DM are used as standard models in velocity-scalar and scalar probability density function methods for reacting flow simulations.[1] (iii) The calculation of $C_0^{(\mathrm{IV})}$ is found to be in excellent agreement with recent results of Sawford and Yeung,[3] which is of relevance as pointed out above. It is shown that the disappearance of u_i^{an} and u_i^{ac} contributions to the variance budget in HIST leads about to the same result as obtained for $C_0^{(\mathrm{III})}$. Consequently, the use of the LM (or corresponding models) to the simulation of flows with a different weight of u_i^{an} and u_i^{ac} contributions in the variance budget requires different C_0 values, i.e., the Kolmogorov constant $C_0(\infty)$ is found to be nonuniversal. It is of interest to compare this conclusion with the opinion of Landau and Lifshitz[16] regarding to the corresponding question related to the Eulerian velocity correlation. On p. 126 they state that the possibility exists in principle to obtain a universal formula for the relation of the corresponding instantaneous variables, i.e., if averaging is not involved. "When we average these expressions, however, an important part will be played by the law of variation of ϵ over times of the order of the periods of the large eddies, and this law is different for different flows. The result of averaging therefore cannot be universal."[16] Contributions due to large-scale eddy motions are represented here by u_i^{an} and u_i^{ac}, which are found to be responsible for the $C_0(\infty)$ variations. Therefore, the results derived here are fully consistent with the view of Landau and Lifshitz.

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