A model for the reduction of the turbulent energy redistribution by compressibility

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Compressibility may strongly reduce the redistribution of turbulent kinetic energy. The question of how such effects can be taken into account in turbulence models is addressed here. First, a corresponding stochastic turbulence model is developed on the basis of a simplification of the generalized Langevin model for turbulent velocities. This model is then reduced to a deterministic model that extends existing methods by the consideration of structural compressibility effects. Combined with stochastic models for scalars, these velocity models may be used for compressible turbulent reacting flow simulations where the consideration of chemical reactions does not require approximations. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1613652]

In general, simulations of turbulent reacting flows require the consideration of compressibility effects. Such effects may be differentiated into two groups: dilatational compressibility effects, which are related to a nonzero mean dilatation or dilatation correlations (the pressure dilatation Π_d and dilatational dissipation ε_d), and structural compressibility effects, which are related to changes of the structure of velocity fields. Available experience seems to indicate that the influence of dilatational compressibility effects is small in flows that do not involve shocks.[1–4] In contrast to this, structural compressibility effects are known to be important.[3] This may be seen by analyzing the asymptotic features of Sarkar's[3] direct numerical simulation (DNS) of compressible turbulent shear flow. To measure the strength of compressibility, Sarkar[3] suggested the use of the gradient Mach number M_g, which is defined by

$$M_g=\frac{Sl_{12}}{\sqrt{\overline{a^2}}}. \tag{1}$$

Here $S=\partial \overline{U}_1/\partial x_2$ is a constant shear rate, a is the instantaneous speed of sound, and l_{12} is the correlation length of u_1 fluctuations in the direction of shear x_2,

$$l_{12}=\frac{1}{\overline{\overline{u_1(x_2)u_1(x_2)}}}\int dr_2\overline{u_1(x_2+r_2)u_1(x_2)}. \tag{2}$$

In these expressions, U_i refers to instantaneous velocities and $u_i=U_i-\overline{U_i}$ to velocity fluctuations. The overbar denotes mass density-weighted ensemble means. Typical asymptotic features of Sarkar's[3] compressible turbulent shear flow DNS are presented in Fig. 1 and Table I. Contributions related to ε_d and $\Pi_d/\langle\rho\rangle$ ($\langle\rho\rangle$ being the ensemble-averaged mass density) are not separately given. However, according to Blaisdell *et al.*,[4] one knows that ε_d and $-\Pi_d/\langle\rho\rangle$ affect the change of turbulent kinetic energy $k=\overline{u_nu_n}/2$ (the sum convention is applied) in the same direction and approximately with the same amount. According to Table I, one may conclude then that $\varepsilon_d/P\approx 0.06$, which implies that $\varepsilon_d/\varepsilon_s\approx 0.12$ and $P/\varepsilon\approx 1.72$. Here, $P=-S\overline{u_1u_2}$ refers to the production of k and $\varepsilon=\varepsilon_d+\varepsilon_s$ to its dissipation rate, where ε_s denotes the solenoidal dissipation. Thus, one finds for the production and dissipation normalized to the characteristic shear flow parameters k and S,

$$\frac{P}{Sk}=-\frac{\overline{u_1u_2}}{k}=0.34\exp(-0.2M_g), \tag{3a}$$

$$\frac{\varepsilon}{Sk}=-\frac{\varepsilon}{P}\frac{\overline{u_1u_2}}{k}=0.20\exp(-0.2M_g). \tag{3b}$$

Hence, the production-to-dissipation ratio P/ε remains approximately unaffected by M_g, but the normalized production P and dissipation ε vanish with growing compressibility (according to the $\overline{u_1u_2}$ curve given in Fig. 1). By introducing the characteristic dissipation time scale $\tau=k/\varepsilon$, we see that the characteristic mixing frequency $(S\tau)^{-1}$ vanishes according to (3b). This implies that the redistribution of turbulent kinetic energy is strongly reduced, such that the features presented in Fig. 1 are found: the turbulence in the x_2 and x_3 directions vanishes and all the turbulent kinetic energy is directed into the streamwise direction x_1. Hence, the shear stress $\overline{u_1u_2}$ must also vanish with growing M_g. A relevant question concerns the physical relevance of M_g; this means whether M_g represents a unambiguous measure for essential flow features or not. By introducing the ratio $r_{32}=\overline{u_3u_3}/\overline{u_2u_2}$ and taking reference to their normalization, $\overline{u_2u_2}$ and $\overline{u_3u_3}$ are given by the relations

$$\frac{\overline{u_2u_2}}{2k}=\frac{1}{1+r_{32}}\left(1-\frac{\overline{u_1u_1}}{2k}\right), \tag{4a}$$

$$\frac{\overline{u_3u_3}}{2k}=\frac{r_{32}}{1+r_{32}}\left(1-\frac{\overline{u_1u_1}}{2k}\right). \tag{4b}$$

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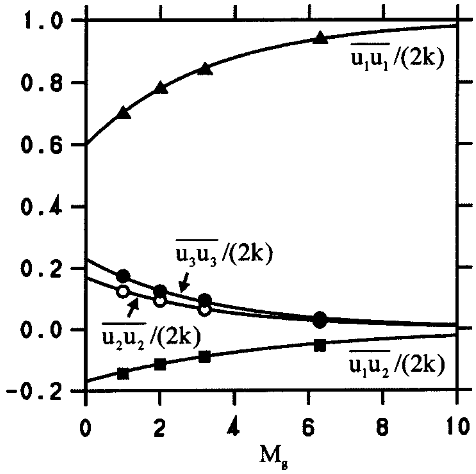

FIG. 1. The symbols present the normalized asymptotic velocity variances of Sarkar's (Ref. 3) A1–A4 compressible shear flow DNS in dependence on the gradient Mach number M_g. The symbol size indicates the variation of these data. The solid lines present corresponding fitting curves that are given in Table I. The Taylor-scale Reynolds number $\mathrm{Re}_\lambda \approx 40$.

The data given in Table I reveal that $r_{32} = 1.35$ independent of M_g. Hence, the diagonal elements of $\overline{u_i u_k}/(2k)$ are controlled by $\overline{u_1 u_1}/(2k)$, which is uniquely related to M_g. Correspondingly, for the flow considered M_g represents a unique measure for the spatial distribution of turbulence. It is worth noting that the concept to parametrize structural compressibility effects in terms of M_g seems to be applicable to a broad range of flows. To clarify this question, one has to study the variation of r_{32} in other flows. For an incompressible equilibrium turbulent boundary layer, for example, one finds $r_{32} = (1.3, 1.5, 1.5)$ for friction Reynolds numbers $\mathrm{Re}_\tau = (180, 395, 590)$, respectively.[5] Further, the influence of compressibility on r_{32} seems to be very small,[6] such that the consideration of a constant r_{32} appears to be an appropriate approximation under many conditions.

A model that involves structural compressibility effects in addition to dilatational compressibility effects in supersonic turbulent reacting flow simulations does not exist at present. For example, Delarue and Pope[7,8] neglected structural compressibility effects in their development of a basic solution for the extension of the applicability of probability density function (PDF) models to compressible reacting flows. However, it is obvious that the consideration of such effects in supersonic turbulent reacting flow simulations may be relevant. To construct such a model, let us consider the generalized Langevin model (GLM) in terms of equations for stochastic realizations. The change of notional particle positions x_i^* and velocities U_i^* is determined in this case by[9]

$$\frac{d}{dt} x_i^* = U_i^*, \tag{5a}$$

$$\frac{d}{dt} U_i^* = -\frac{1}{\langle \rho \rangle} \frac{\partial \langle p \rangle}{\partial x_i} + G_{ik}(U_k^* - \overline{U}_k) + \sqrt{C_0 \varepsilon} \frac{dW_i}{dt}. \tag{5b}$$

Here $\langle p \rangle$ denotes the ensemble-averaged pressure p, G_{ik} is an unknown coefficient matrix, and C_0 is a model parameter. In addition, dW_i/dt refers to a Gaussian process with vanishing means, $\langle dW_i/dt \rangle = 0$, and uncorrelated values at different times, $\langle dW_i/dt(t) \cdot dW_j/dt'(t') \rangle = \delta_{ij}\delta(t-t')$. Here, $\delta(t-t')$ is the delta function and δ_{ij} is the Kronecker delta. A detailed discussion of the structure of Eqs. (5a)–(5b) may be found elsewhere.[9–11] To apply (5a)–(5b), one has to parametrize G_{ik} and C_0. In accord with recent findings presented by the author,[5] one may set $C_0 = 2.0$. To calculate G_{ik}, one may adopt the relationship between stochastic Lagrangian turbulence models and transport equations for Reynolds stresses.[12–14] In particular, one may use algebraic equations for the normalized Reynolds stresses $\overline{u_i u_k}/(2k)$, which follow from (5a)–(5b). This results in six relations for the nine components of G_{ik}. A unique relationship between the components of the Reynolds stress tensor and G_{ik} is obtained by assuming G_{ik} to be symmetric as the Reynolds stress tensor. This results in an extended Langevin model (ELM).[11] For the flow considered, the diagonal elements of G_{ik} read as

$$\frac{\overline{u_1 u_1}}{2k}\left(-2G_{11}\tau + \frac{P}{\varepsilon} - 1 + \frac{\Pi_d}{\langle \rho \rangle \varepsilon}\right) = \frac{1}{2}C_0 + \frac{P}{\varepsilon} + \frac{\overline{u_1 u_2}}{k} G_{12}\tau, \tag{6a}$$

$$\frac{\overline{u_2 u_2}}{2k}\left(-2G_{22}\tau + \frac{P}{\varepsilon} - 1 + \frac{\Pi_d}{\langle \rho \rangle \varepsilon}\right) = \frac{1}{2}C_0 + \frac{\overline{u_1 u_2}}{k} G_{12}\tau, \tag{6b}$$

$$\frac{\overline{u_3 u_3}}{2k}\left(-2G_{33}\tau + \frac{P}{\varepsilon} - 1 + \frac{\Pi_d}{\langle \rho \rangle \varepsilon}\right) = \frac{1}{2}C_0, \tag{6c}$$

and the off-diagonal components of G_{ik} are given by

$$\frac{\overline{u_1 u_1}}{k} G_{12}\tau\left(1 + \frac{\overline{u_1 u_1}\;\overline{u_2 u_2}}{\overline{u_1 u_2}^{\,2}}\right) = \frac{1}{2}C_0 + \frac{P}{\varepsilon}\left(1 - \frac{\overline{u_1 u_1}\;\overline{u_2 u_2}}{\overline{u_1 u_2}^{\,2}}\right)\left(1 + \frac{\overline{u_1 u_1}}{\overline{u_2 u_2}}\right)^{-1}, \tag{7a}$$

$$G_{13} = 0, \tag{7b}$$

$$G_{23} = 0. \tag{7c}$$

It is worth noting that the ELM has another structure than the Haworth–Pope model[12] for G_{ik}. In contrast to the ELM, the

TABLE I. The analytical features of the asymptotic statistics of Sarkar's (Ref. 3) A1–A4 compressible turbulent shear flow DNS. The normalized variances are given as fitting curves to Sarkar's (Ref. 3) DNS data; see Fig. 1. For the flow considered, $\overline{u_1 u_3} = \overline{u_2 u_3} = 0$.

Variances	Production-to-dissipation ratio
$\frac{\overline{u_1 u_1}}{2k} = 1 - 0.40\,\exp(-0.3M_g) \pm 0.01$	$\frac{\varepsilon_s}{P} = 0.52 \pm 0.04$
$\frac{\overline{u_2 u_2}}{2k} = 0.17\,\exp(-0.3M_g) \pm 0.01$	$\frac{\varepsilon_s}{P} + \frac{\varepsilon_d}{P} - \frac{\Pi_d}{\langle \rho \rangle P} = 0.64 \pm 0.02$
$\frac{\overline{u_1 u_2}}{2k} = -0.17\,\exp(-0.2M_g) \pm 0.005$	

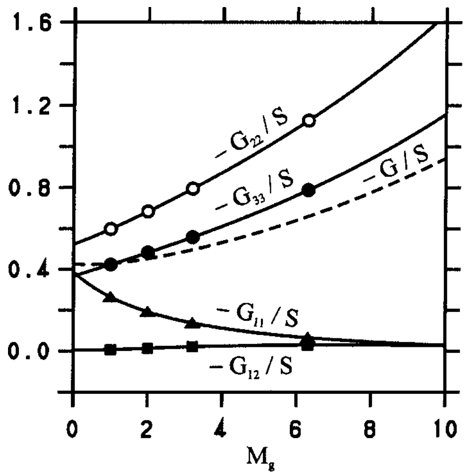

FIG. 2. The ELM coefficients normalized to S are shown in dependence on the gradient Mach number M_g. For this, the relations (6a)–(6c) and (7a) are used in combination with $C_0=2.0$ and the data presented in Table I. The symbols present Sarkar's (Ref. 3) DNS data. The dashed line shows $G/S = (G_{11}+G_{22}+G_{33})/(3\text{ S})$ for a comparison.

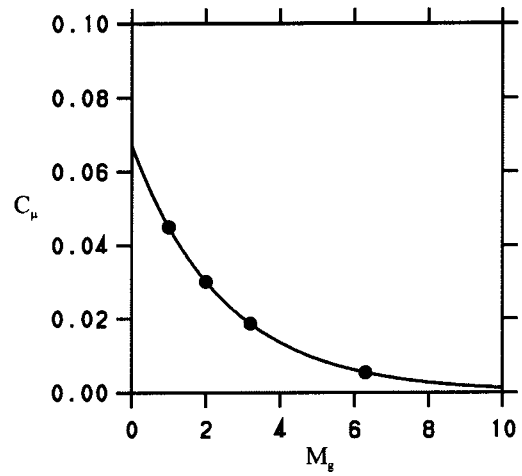

FIG. 3. The solid line presents C_μ in dependence on the gradient Mach number M_g according to relation (11). The symbols show the corresponding DNS data of Sarkar (Ref. 3).

Haworth–Pope model makes use of the assumption that G_{ik} is proportional to the normalized anisotropy tensor $\overline{u_i u_k}/(2k) - \delta_{ik}/3$. As may be seen by means of (6a)–(6c), this proportionality assumption corresponds with the consideration of small anisotropy effects within the frame of the Haworth–Pope model.

The ELM can be applied to three-dimensional, inhomogeneous compressible turbulent flows. As an example, calculations of G_{ik} according to the relations (6a)–(6c) and (7a) are presented in Fig. 2 for the compressible turbulent shear flow considered by Sarkar.[3] For an incompressible flow ($M_g=0$), the coefficients G_{11}, G_{22}, and G_{33} have comparable values $(G_{11}, G_{22}, G_{33}) = -(0.39, 0.52, 0.37)$ S. This finding is well supported by corresponding results for an equilibrium turbulent boundary layer:[5] for a friction Reynolds number $\mathrm{Re}_\tau=590$, for example, one finds $(G_{11}, G_{22}, G_{33}) = -(0.51, 0.76, 0.49)$ S (the difference to the corresponding values presented above may be attributed to the fact that $P=\varepsilon$ for the equilibrium turbulent boundary layer). However, G_{11}, G_{22}, and G_{33} show a different behavior with growing compressibility: G_{11} vanishes whereas G_{22} and G_{33} go to minus infinity (in a ratio $G_{33}/G_{22}=0.7$, the deviations from this value are below 1%). A relevant finding is that G_{12} obviously vanishes (its maximum is $G_{12} = -0.029$ S). This behavior is fully consistent with the results of a corresponding study for incompressible flows.[5] One finds that the Langevin model (LM), where only the diagonal components of G_{ik} are nonzero, represents a sufficiently general model for the flow considered. The latter provides further evidence for the suitability of the value chosen for C_0 here.[5] The fact that G_{12} is found to be vanishingly small also for compressible flow (for nonzero M_g) is plausible: the reduction of the energy redistribution by compressibility cannot be expected to produce a direct dependence of the dynamics of u_1 fluctuations on u_2 fluctuations. Compared to existing methods, it is worth noting that a reduction of the ELM to the simplified Langevin model (SLM) with an isotropic coefficient matrix $G_{ik}=G\,\delta_{ik}$ does not enable accurate flow simulations. As may be seen in terms of Fig. 2, the approximation $G_{ik}=G\,\delta_{ik}$ is related to significant errors regarding the modeling of G_{22} and G_{33}, and a qualitatively wrong behavior of G_{11}.

A simpler approach than the use of the stochastic model (5a)–(5b) for compressible reacting flow simulations is given by reducing (5a)–(5b) to a model for mean velocities. This model reads as

$$\frac{\partial \overline{U}_i}{\partial t} + \overline{U}_k \frac{\partial \overline{U}_i}{\partial x_k} = -\langle \rho \rangle^{-1} \frac{\partial \langle \rho \rangle \overline{u_i u_k}}{\partial x_k} - \langle \rho \rangle^{-1} \frac{\partial \langle p \rangle}{\partial x_i}. \tag{8}$$

To close (8), one can apply (5a)–(5b) to derive a model for the Reynolds stress tensor. Very often, such models are further simplified to algebraic approximations for the anisotropic part of the Reynolds stress tensor. Such an expression can be derived as a consequence of (5a)–(5b). This results in[9,11]

$$\overline{u_i u_k} = \frac{2k}{3}\,\delta_{ik} - 2\,\nu_T \overline{S}_{ik}^d, \tag{9}$$

where the deviatoric part $\overline{S}_{ik}^d$ of the rate-of-strain tensor is given by

$$\overline{S}_{ik}^d = \frac{1}{2}\left(\frac{\partial \overline{U}_i}{\partial x_k} + \frac{\partial \overline{U}_k}{\partial x_i} - \frac{2}{3}\,\frac{\partial \overline{U}_n}{\partial x_n}\,\delta_{ik} \right), \tag{10}$$

and the turbulent viscosity by $\nu_T = C_\mu k \tau$.

In correspondence to the application of the ELM to the compressible turbulent shear flow considered by Sarkar,[3] we use (9) for the same flow in order to calculate C_μ. The production of turbulent kinetic energy is then given by $P = C_\mu k S^2 \tau$. By dividing this expression by $\varepsilon = k/\tau$, we find the relation $P/\varepsilon = C_\mu S^2 \tau^2$ for the calculation of C_μ. The use of $P/\varepsilon \approx 1.72$ and (3b) for $S\tau$ then results in

$$C_\mu = 0.07\,\exp(-0.4 M_g). \tag{11}$$

This curve (11) is shown in Fig. 3 together with the corresponding DNS data of Sarkar.[3] The value $C_\mu = 0.07$ at M_g

$=0$ is in accord with $C_\mu = 0.08 \pm 0.01$, which was found by the DNS of incompressible flows; see Pope, p. 371.[9] The most relevant finding presented in Fig. 3 is the significant reduction of C_μ by compressibility, which assures the corresponding reduction of the production of k. Obviously, the neglect of this reduction in methods applied usually[1] may be related to significant errors.

The findings obtained may be summarized in the following way. (i) The gradient Mach number M_g was shown to represent a unique measure for the spatial distribution of turbulence (the three diagonal components of the normalized Reynolds stress tensor) provided the variance ratio r_{32} is constant. As a result of the parametrizations of velocity variances presented in Table I, it was shown that this condition is accurately satisfied for the compressible turbulent shear flow considered: one finds $r_{32}=1.35$ independent of M_g. Further, it was pointed out that the use of a constant r_{32} appears to be an appropriate assumption with regard to the modeling of many turbulent flows. (ii) The generalized Langevin model for turbulent velocities was reduced to a stochastic turbulence model that takes (in addition to dilatational compressibility effects) structural compressibility effects into account. The structure of this model is basically the same as for incompressible flows (only diagonal elements of G_{ik} are nonzero). Its agreement with the LM provides further evidence for the suitability of the value chosen for C_0 here. Further, it was shown that compressibility may have strong effects on turbulent mixing frequencies; see Fig. 2. These effects cannot be covered on the basis of the SLM applied previously. The assumption $G_{ik}=G\delta_{ik}$ may result in significant errors regarding the modeling of G_{22} and G_{33}, and a qualitatively wrong behavior of G_{11}. (iii) Instead of using stochastic turbulence models, compressible flows are often simulated on the basis of deterministic models, as given by the mean velocity equation (8) combined with (9) for Reynolds stresses and transport equations for k and ε.[1] The reduction of the stochastic turbulence model presented here to such a deterministic model reveals that the significant reduction of the production of turbulent kinetic energy can be simulated in this way. However, this requires (in contrast to methods applied usually[1]) the variation of C_μ as function of M_g according to relation (11).

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